Numerical Analysis

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Lecture – 01
Introduction

Hi, welcome to this NPTEL course on Numerical Analysis. This is our first lecture. In this lecture, I will introduce you the subject Numerical Analysis and also I will give you some motivations for why we have to learn the subject as a course and then I will quickly go through some of the mathematical preliminaries that are needed for us to follow the course.

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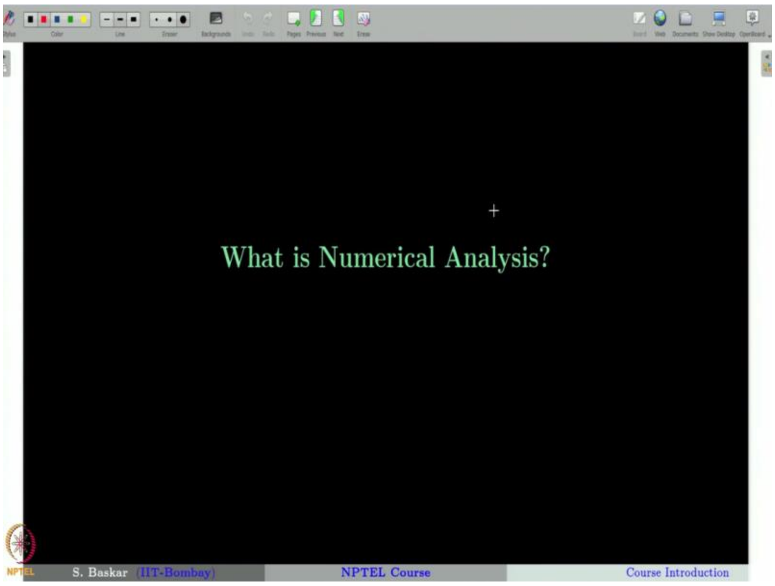

Let us get into the first question of what is numerical analysis?

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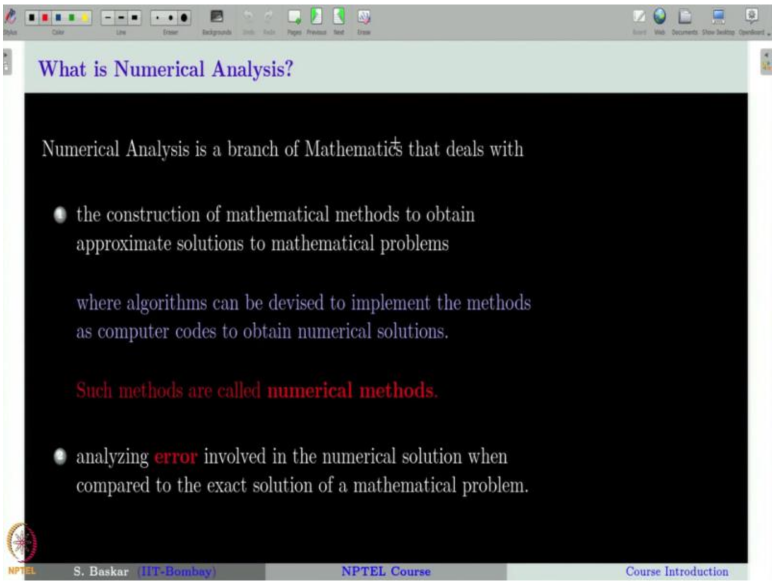

Well, Numerical Analysis is a branch of Mathematics that deals with the construction of mathematical methods to obtain approximate solutions to some mathematical problems and this mathematical methods can also be written in the form of an algorithm that can be implemented as computer course to generate numerical solutions. In short, Numerical Analysis deals with constructing numerical methods for some mathematical problems.

Once we know that the numerical methods are going to generate only approximate solutions, the next natural question is, what is the error involved in the numerical solutions when compared to the exact solution of the mathematical problem. This leads to error analysis of the methods that we develop. Therefore, Numerical Analysis has two components. One is to develop the numerical methods for some mathematical problems and the next component is to do the error analysis for this methods.

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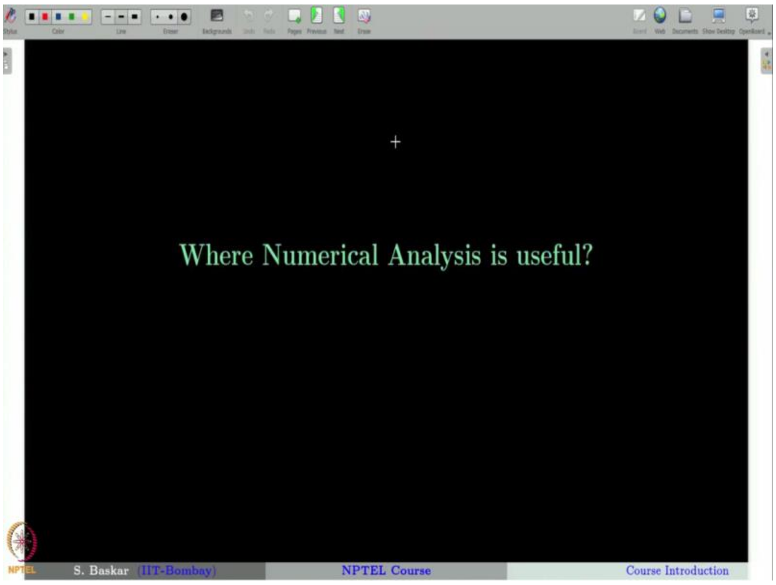

Now the next question where Numerical Analysis is useful? Let me just give you a broad idea of where numerical approximations are used.

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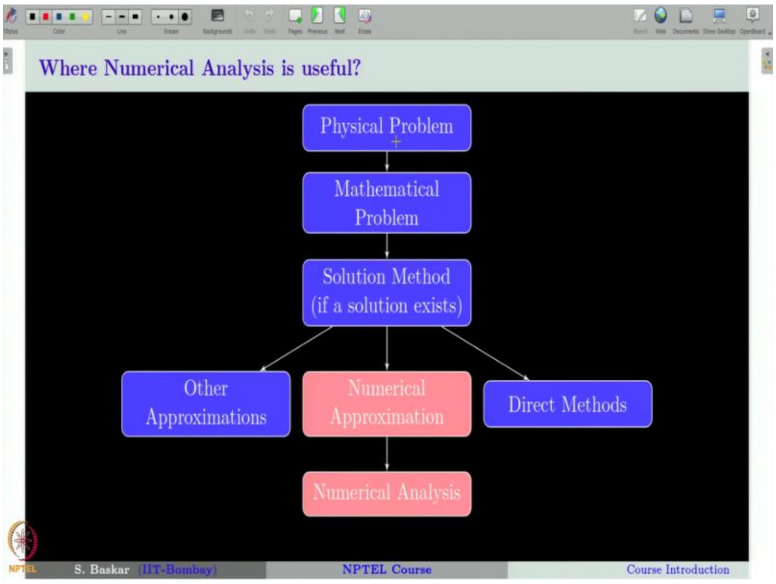

Let us assume that we have a physical phenomenon that we want to understand. One way to do that is to perform some experiments on the problem. Another way is to develop a mathematical problem in the sense that we have a mathematical model that can mimic to some extent the physical phenomena. Once we have such a mathematical problem then the next question is how to solve this mathematical problem.

Generally, we can have methods that can give exact solutions to this mathematical problems, but such direct methods are quite rare in the sense that if the mathematical problem is very simple we can have such methods, but in general if the physical phenomena is highly non linear then the corresponding mathematical problem will be quite complicated to solve exactly. Therefore, one needs to go for approximations. When you talk about approximations there are two levels of approximations. One is the approximation to the model itself something like you have non linear problem then you linearize the problem, solve it and somehow try to recover the non linearity later on that is something like making the approximation to the model itself. And another approach is to device some numerical methods to get approximate solution to our mathematical problems. So, the second one is what we call as numerical approach. Once we are into this numerical approach then we are into the Numerical Analysis.

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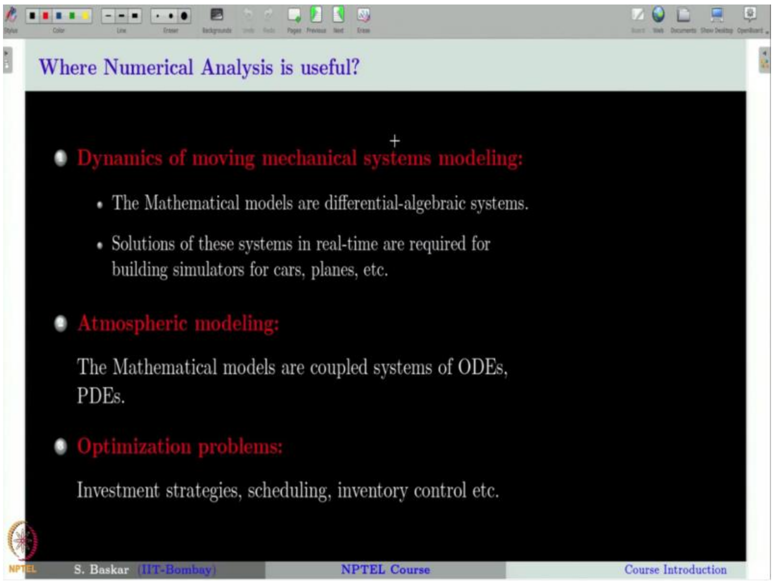

There are various problems in applications that are quite difficult to solve exactly, let me list very few here. First is, dynamics of moving mechanical systems model. In this the mathematical models are generally systems of differential and algebraic equations and the solutions of these systems in real time are required for building simulators for cars, planes etc. Another example is atmospheric modeling. The mathematical models that come in atmospheric modeling are coupled systems of ordinary and partial differential equations and the third example is the optimization problems which comes in investment, strategies, scheduling inventory controls etc.

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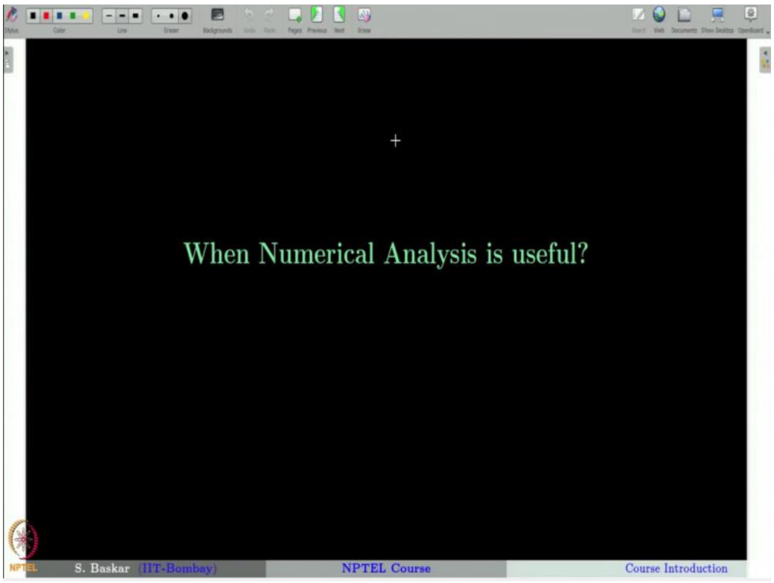

Then next question is when Numerical Analysis is useful?

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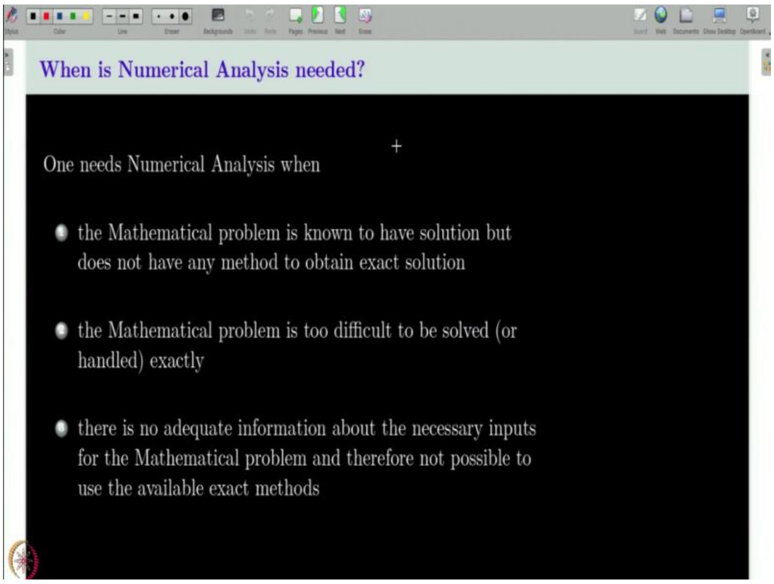

Well, one needs Numerical Analysis when the mathematical problem is known to have solution, but does not have a method that can give exact solutions. Sometimes the mathematical problem is too difficult to be solved exactly and there are also situations where we do not have adequate information about the necessary inputs of the mathematical problem.

And therefore it is not possible to use the available exact methods. These are some situations where we have to go for certain approximations. As I told if we chose to go for numerical approximations then Numerical Analysis has to be understood thoroughly.

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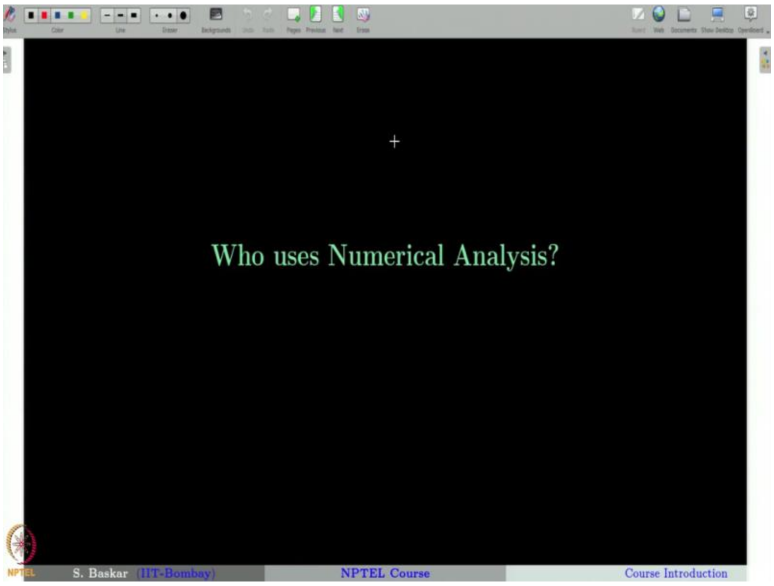

Who uses Numerical Analysis?

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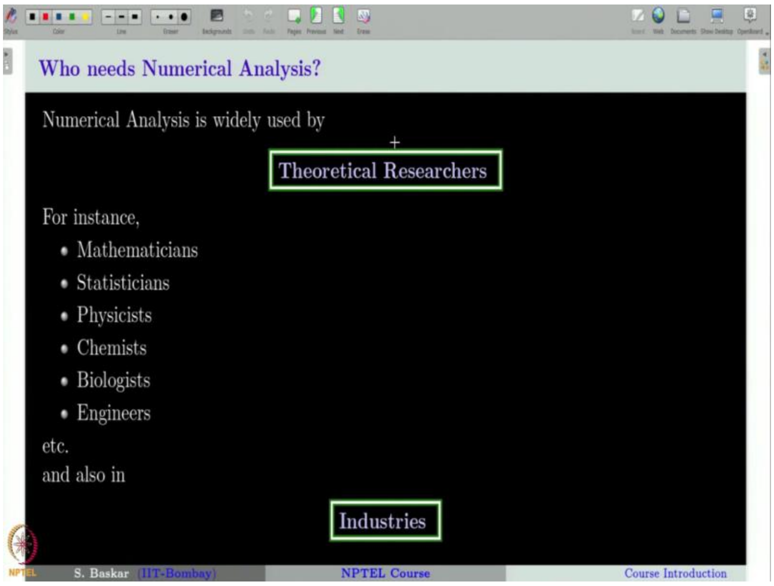

Well, almost all fields of science and engineering to list few mathematicians, statistician, physicist, chemist, biologist and almost all fields of engineering people use Numerical Analysis and nowadays even in industries people use software that in turn runs many numerical methods to generate certain simulations.

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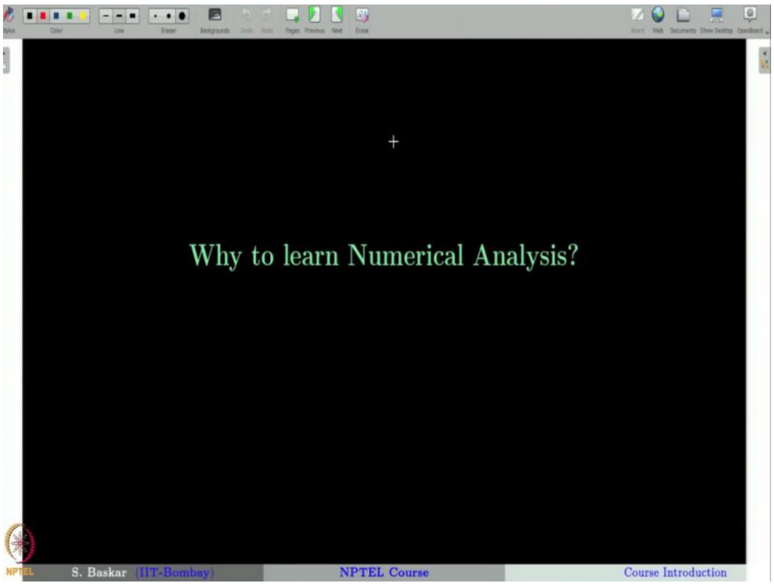

Our next question is why to learn Numerical Analysis?

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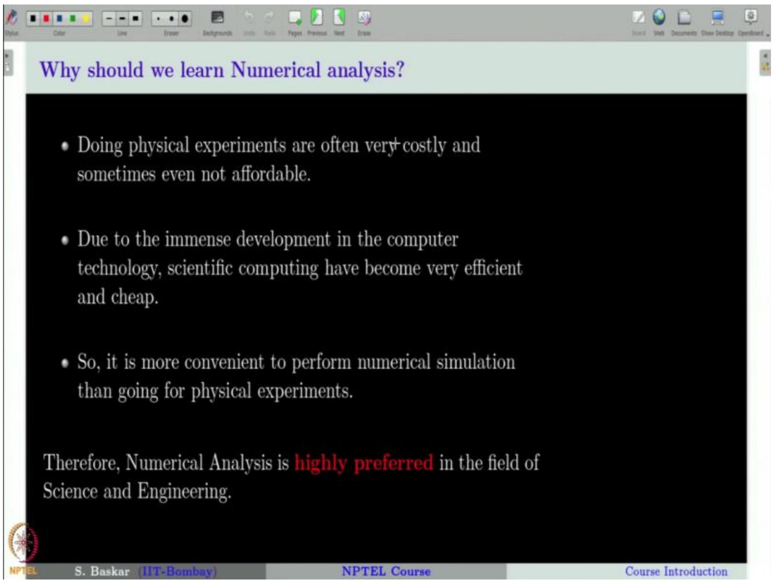

Well, as I told you to understand certain physical problems one way to do is to perform experiments on this physical problems, but often physical experiments are very costly and even sometimes not affordable. Due to the immense development in the computer technology, Scientific computing have become very efficient and cheap. So, it is more convenient to perform numerical simulations than going for physical experiments quite often. Therefore, Numerical Analysis is highly preferred in the field of science and engineering.

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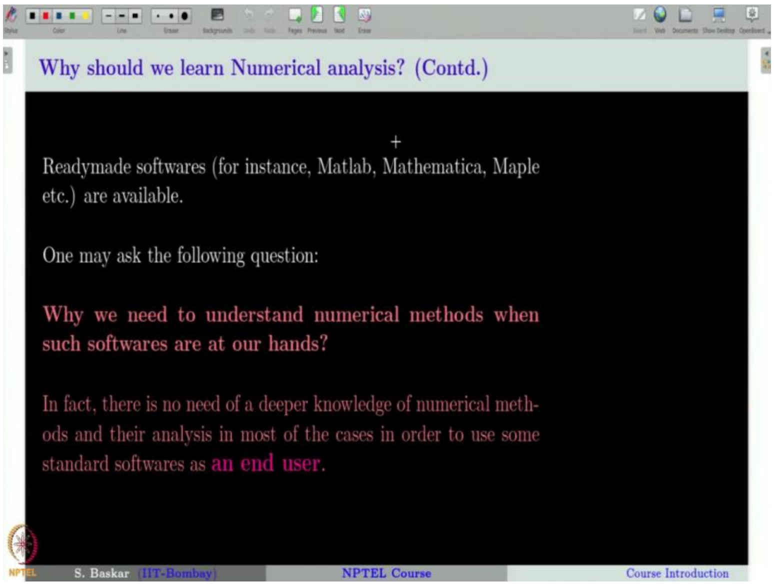

Well, there is another question which generally students ask. We know that there are readily available softwares like Matlab, Mathematica, Maple and so on where you can simply use certain simple commands to perform all well-known numerical methods. In fact, there is no need of a deeper knowledge of these numerical methods and their analysis in most of the cases in order to use some such softwares as end users.

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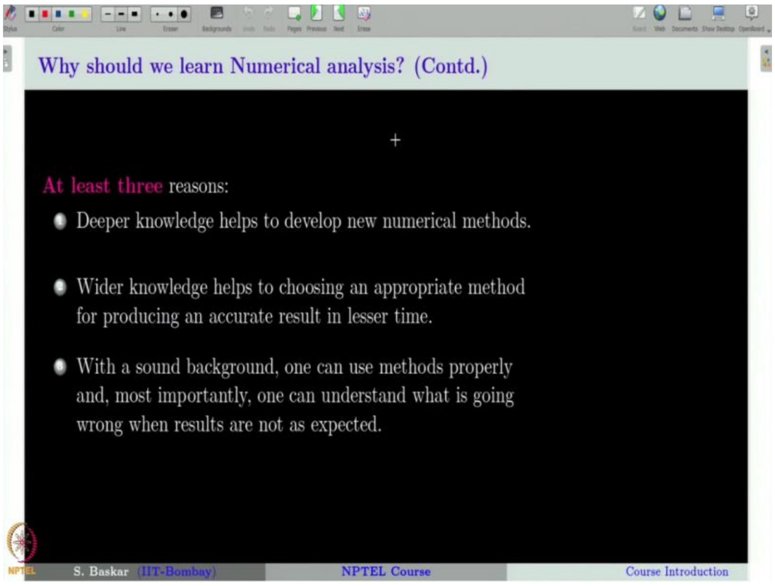

However, we have at least three reasons for why we have to understand these numerical methods from their creators point of view. Let me list them. One is a deeper understanding of these methods will help us to develop new numerical methods especially when the available ones are not suitable for our problem. Wider knowledge of numerical methods and their analysis will certainly help us to choose an appropriate method for producing an accurate result in lesser time.

This is especially useful when we have more than one method for a given mathematical problem. Of course, with a sound background one can use methods properly and most importantly one can understand what is going wrong when the results are not as expected. Therefore, it is very important for us to understand certain numerical methods and their analysis before actually going into use them.

Well, especially we will try to learn some mathematical tools to do error analysis. From this point of view we will consider some simple mathematical problems.

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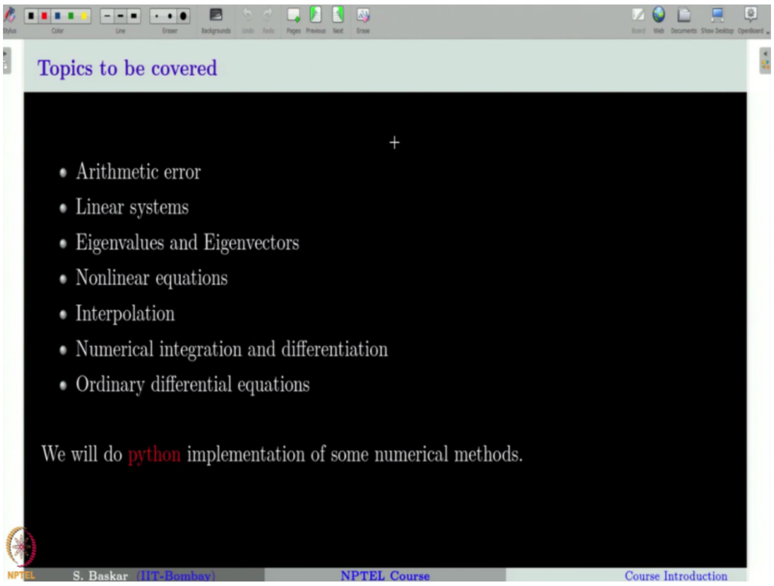

We will start our course with arithmetic errors. Well, we will see in detail what it is. Then we will continue with the first mathematical problem of solving linear systems and then we will go to develop methods for computing eigen values and eigen vectors of a given matrix and then we will see some iterative methods to capture simple roots of a given non linear equations. Then we learn polynomial interpolation of a continuous function. Then we will learn how to develop quadrature formulas.

These are the numerical formulas for integration and also, we will learn to develop finite difference formula which are formulas for differentiation of a function. Finally, we will also learn to develop some numerical methods for differential equations. It is also important for a Numerical Analyst to know some computer coding, so we will also learn Python implementation of some simple numerical methods.

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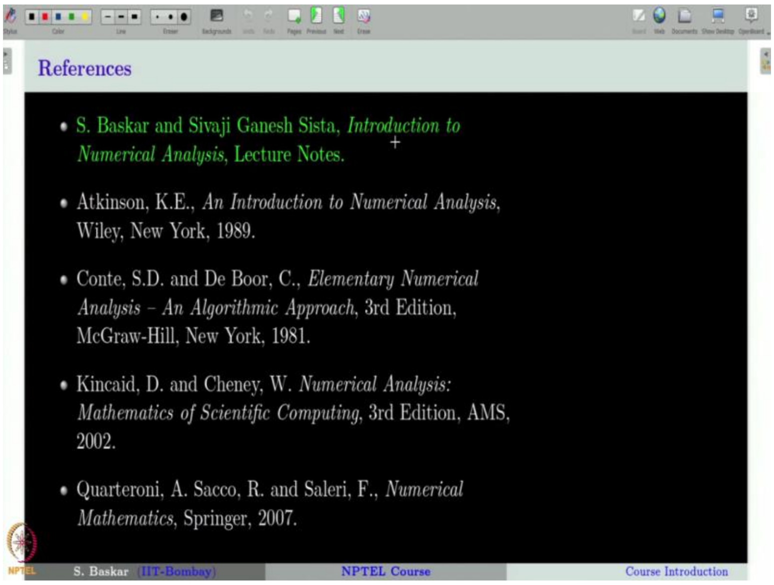

Well, I with my colleagues Sivaji Ganesh have developed a lecture notes on the subject. I will be mostly following our lecture notes. I will also give you PDF files of this lecture notes. Apart from this we also have some books. \In fact we have used these books in developing our lecture notes. These are Atkinson's Introduction to Numerical Analysis, Conte De Boor classical book on Elementary Numerical Analysis an Algorithmic Approach.

We also used Kincaid and Cheney Numerical Analysis and Quarteroni et al. Numerical Mathematics. Let us quickly review the results that are often used in our course.

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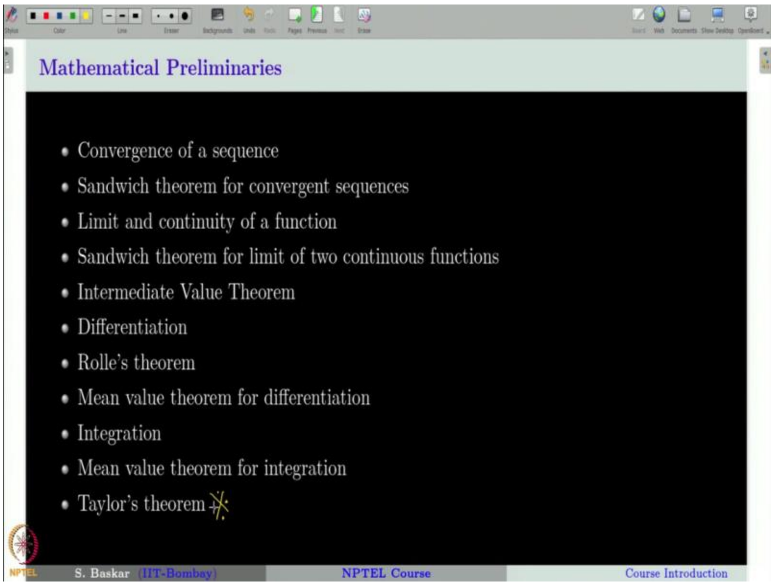

The methods that we use are basically the iterative methods and their outcomes are basically sequence of numbers or functions and therefore we need to know the convergence of a sequence. In particular quite often we use very interesting result called sandwich theorem for convergent sequences. Let me quickly recall what is this theorem. Suppose, you have three sequences a_n, b_n and c_n.

If we know that if $a_n \leq b_n \leq c_n$ at least for sufficiently large n's that is you can find a $N \geq 0$ such that this holds for all $n \geq N$ and further if we know that a_n is converging to L and c_n is also converging to L as $n \to \infty$, the same limit, then it is quite intuitive to see that b_n will also converge to L as $n \to \infty$, that is what the sandwich theorem says.

The next is limit and continuity of a function and then we also have sandwich theorem for limit of continuous functions. The idea is exactly the same as in the convergence sequence. Suppose, you have three functions f, g and h. Suppose, we know that $f \leq g \leq h$ say for all x in a small neighborhood of x_0 and we know that f is converging to L say for instance when x is converging to x_0. So, all these are happening that is this condition happens at a small neighborhood of x_0 say $(x_0 - \delta, x_0 + \delta)$ and this is also converging to L as $x \to x_0$ then we can say that g also converges to L as $x \to x_0$. So, the idea is exactly the same as in the convergence sequence. So, these are the results we will be often using in the non linear equations chapter.

Then comes the intermediate value theorem which says that you have a function which is continuous and suppose you know that at a and b the function takes the value say x and y. Then you give any N in between x and y you can always find a ξ in the interval (a, b) such that $f(\xi) = N$. This holds when f is a continuous function that is also intuitively clear and the next topic that we will be using is the differentiation. Of course as I have told we will develop numerical methods to approximate derivatives of a function. Therefore, we need to know the basic theory of differentiation of a function.

And in connection to it some of the important theorems like Rolle's Theorem which says that the value of f at the point a and b are equal say and if f is a C^1 function it means the function f is continuously differentiable in the interval $[a, b]$ then Rolle's Theorem says that you can always find a point in the interval $[a, b]$ say something ξ such that $f'(\xi) = 0$ that is also intuitively very clear.

You know that $f(a)$ is equals to $f(b)$ and the function is basically a continuous function. Therefore, it has to take a turn something like u-turn to come back to the same place as it started, but then since it is also C^1 therefore any point at which it turns it is going to turn where its first derivative is also a continuous function. Therefore, you have f' exist at all points in the interval (a, b). And since it takes a turn, you obviously will have some point at which f' is 0.

The next theorem is the mean value theorem for differentiation. It is a well known theorem and it is also important that we will use it quite often in our course. The theorem says that suppose we have a C^1 function defined on interval $[a, b]$ then we can write $f(a) - f(b) = f'(\xi)(a - b)$for some ξ lies between a and b.

So, that is the statement of the theorem. What is the physical interpretation of this? Well, you have the interval (a, b) and suppose your function takes the value $f(a)$ at a and $f(b)$ at b and suppose the graph of the function is something like this. Assume that it is a nice function and now you take the secant line joining the point $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$, this is the secant line, and its slope is given by $\frac{(f(a)-f(b))}{(a-b)}$.

So, the theorem says that you can find a ξ in between the point a and b such that the slope of the tangent line, something like this, may be ξ is here and that is parallel to the secant line that is what the theorem says. So, you can find a ξ such that this tangent line whose slope of $f'(\xi)$ is parallel to the secant line. So, this can be written as $\frac{(f(a)-f(b))}{(a-b)}$, that is the slope of this secant line. And that is equal to $f'(\xi)$ which is the slope of this tangent line.

Then next is that we should know the theory of integration and also an important theorem in this chapter that is the mean value theorem for integration which states that suppose we have two functions f and g defined on the interval $[a, b]$ and given that g is of one sign say $g > 0$ or $g < 0$ on the interval $[a, b]$.

Then you can write $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some ξ in the interval (a, b). This also used in our course quite often especially when we are doing the error analysis for integration and so on. The next theorem is the well known Taylor's Theorem. We can say that this Taylor's Theorem is the heart of our course because it is used almost in all the chapters other than the linear algebra chapter.

Therefore, understanding Taylor's Theorem is very, very important for us and after this we will also need the notation called Big Oh and Small oh. These notations are to understand the speed of convergence of a convergence sequence and that also leads to the concept of order of convergence which is also very important for our course. Therefore, in the next two lectures we will go little detail into these two topics that is Taylor's Theorem and order of convergence and see how these theorems and concepts can be used in Numerical Analysis. In our notes we will give a appendix where we will summarize all these topics and we will also give some exercise problems. You go through this appendix so that you can brush up your calculus and that will be easier for you to follow the course and also you solve all those problems so that you will get more familiarity with all this topics. Thanks for your attention.