Fourier Analysis and its Applications

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09 Parseval formula. Isoperimetric theorem

Theorem (Bessel's inequality): If a_0, a_n, b_n $(n = 1, 2, \dots)$ are the Fourier coefficients of a function $f \in L^2[-\pi, \pi]$ we have the following inequality:

$$|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx \tag{2.4}$$

Proof: We have already seen that $1, \cos x, \sin x, \dots, \cos Nx, \sin Nx$ are orthogonal to $f(x) - S_N(f, x)$ whereby we conclude their linear combination $S_N(f, x)$ is also orthogonal to $f(x) - S_N(f, x)$. The Pythagorous identity now gives:

$$\|f(x) - S_N(f, x)\|^2 + \|S_N(f, x)\|^2 = \|f\|^2 \tag{2.5}$$

Hence

$$\|S_N(f, x)\|^2 \leq \|f\|^2 \tag{2.6}$$

We now compute

$$\begin{aligned} \|S_N(f, x)\|^2 &= \int_{-\pi}^{\pi}|a_0 + \sum_{j=1}^{N}(a_j \cos jx + b_j \sin jx)|^2dx \\ &= 2\pi|a_0|^2 + \pi\sum_{j=1}^{N}(|a_j|^2 + |b_j|^2) \end{aligned}$$

So the inequality $\|S_N(f, x)\|^2 \leq \|f\|^2$ translates to

$$|a_0|^2 + \frac{1}{2}\sum_{n=1}^{N}(|a_n|^2 + |b_n|^2) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx$$

Letting $N \longrightarrow \infty$ we get the result.

The Parseval formula It turns out that the Bessel's inequality is actually an equality!

Theorem: Suppose $f \in L^2[-\pi, \pi]$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2). \tag{2.7}$$

Formula (2.7) has a simple physical interpretation. If we think of $f(x)$ as a $2\pi-$periodic signal, the left hand side represents the *energy of the signal* while the right hand side spells out the *contribution from each of the Fourier components* of the signal.

Proof of (2.7) is technical and we shall return to it later after proving *Fejer's theorem.* Instead we look at a couple of simple examples.

Examples on Parseval Formula: Consider the function $f(x) = x^2$ on $[-\pi, \pi]$ extended over $\mathbb{R}$ as a $2\pi-$periodic function. Let us compute the Fourier coefficients of this function.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$$

Obviously $b_n = 0$ and we have for a_n,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \frac{2}{n\pi}\int_0^{\pi} x^2 \frac{d(\sin nx)}{dx} dx = \frac{-4}{n\pi}\int_0^{\pi} x \sin nx dx$$

One more integration by parts gives:

$$a_n = \frac{4(-1)^n}{n^2}$$

Parseval's formula now gives:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

which after simplification gives the formula

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

If you have worked out the exercises in the previous module you would have obtained

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Exercise: Show that $\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

Bernoulli polynomials and Bernoulli numbers To describe the values of the zeta function $\zeta(2k)$ for $k = 1, 2, 3, \ldots$ we need to introduce the *Bernoulli numbers.* Define inductively the sequence of polynomials $B_n(x)$ as

$$\begin{aligned} B_0(x) &= 1 \\ B_n'(x) &= nB_{n-1}(x), \quad n \geq 1 \\ \int_0^1 B_n(x)dx &= 0 \end{aligned}$$

The polynomials $B_n(x)$ are called the *Bernoulli polynomials.*

Exercise: Calculate the first few Bernoulli polynomials. The numbers $B_n(0)$ are called *Bernoulli numbers.*

Some classical formulas (James Bernoulli - Ars Conjectandi (1713)):

$$1^p + 2^p + \cdots + n^p = \frac{1}{n+1}(B_{p+1}(n+1) - B_{p+1}(0)), \quad p = 1, 2, 3, \ldots$$

L. Euler:

$$\zeta(2p) = 1 + \frac{1}{2^{2p}} + \frac{1}{3^{2p}} + \cdots = \frac{(-1)^{p+1}(2\pi)^{2p} B_{2p}(0)}{2 \cdot (2p)!}, \quad p = 1, 2, 3, \ldots \tag{2.8}$$

James and John Bernoulli tried in vain to obtain the latter for p = 1. It was discovered by Euler in 1736. However James Bernoulli did not live to see the last displayed formula in which the numbers that bear his name feature so prominently.

Exercise: Use $x(\pi^2 - x^2)$ to find the value of

$$\zeta(6) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$$

Is your result in agreement with (2.8)?

Descarte's isoperimetric problem

This is one more important classic variational principle which goes back at least to *René Descartes:*

Theorem: Of all piecewise smooth closed curves with a *given perimeter* the circle encloses *maximum area.*

One can turn the problem around by fixing the area and minimizing the perimeter.

Theorem: Of all piecewise smooth closed curves enclosing a *given area* the circle has the *least perimeter.*

The theorem generalizes to higher dimensions in an obvious way.

Spacial isoperimetric theorem *"With a little knowledge of the physics of surface tension, we could learn the isoperimetric theorem from a soap bubble.*

Yet even if we are ignorant of serious physics, we can be led to the isoperimetric theorem by quite primitive considerations. We can learn it from a cat. I think you have seen what a cat does when he prepares himself for sleeping through a cold night: he pulls in his legs, curls up, and, in short, makes his body as spherical as possible. He does so obviously, to keep warm, to minimize the heat escaping through the surface of his body. The cat who has no intension of decreasing his volume, tries to decrease his surface. He solves the problem of a body with a given volume and minimum surface in making himself as spherical as possible. He seems to have some knowledge of the isoperimetric theorem." Quotation from p. 170 of *G. Polya, Mathematics and plausible reasoning, Princeton University Press, Princeton, 1954.*

Theorem (Hurwitz' Proof of isoperimetric theorem (1902)): Let the piecewise smooth closed curve be parametrized by arc-length $(x(s), y(s))$, $0 \leq s \leq L$ and the curve is traced counter-clockwise. The area A is given by

$$A = \oint x dy = \int_0^L x \frac{dy}{ds} ds \qquad (How?)$$

Let $t = (2\pi s/L) - \pi$ so that t runs over the interval $[-\pi, \pi]$. Then

$$A = \int_{-\pi}^{\pi} x \frac{dy}{dt} dt \tag{2.9}$$

For the perimeter we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \frac{L^2}{4\pi^2}.$$

which is conveniently rewritten as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\} dt = \frac{L^2}{4\pi^2}. \tag{2.10}$$

We now apply the Parseval's formula to (2.9) and (2.10). Let the $n-$th Fourier coefficients of $x(t)$ be a_n, b_n and those of $y(t)$ be c_n, d_n. For the area integral we get

$$A = \pi \sum_{n=1}^{\infty} n(a_n d_n - b_n c_n).$$

For second,

$$L^2 = 2\pi^2 \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2)$$

Thus we see

$$\begin{aligned} L^2 - 4\pi A &= 2\pi^2 \sum_{n=1}^{\infty} \left\{ n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2) - 2n(a_n d_n - b_n c_n) \right\} \\ &= 2\pi^2 \sum_{n=1}^{\infty} \left\{ (na_n - d_n)^2 + (nb_n + c_n)^2 + (n^2 - 1)(c_n^2 + d_n^2) \right\} \end{aligned}$$

Thus

$$L^2 \geq 4\pi A$$

and the maximum value of the enclosed area equals $L^2/4\pi$. To determine the curve that achieves this, equality must hold which is so if and only if

$$na_n - d_n = nb_n + c_n = c_n = d_n = 0, \quad n = 2, 3, \ldots$$

and $a_1 = d_1, b_1 = -c_1$. Thus

$$x(t) = a_0 + a_1 \cos t + b_1 \sin t, \; y(t) = c_0 - b_1 \cos t + a_1 \sin t.$$

which represents a circle. The proof is complete.