Fourier Analysis and its Applications

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08 Least square approximation

In this chapter we shall discuss two themes:

(1) Convergence in mean. We shall introduce the appropriate function space and cover the requisite preliminaries. We shall discuss two applications of the *Parseval formula:*

- *Hurwitz's* proof of the *isoperimetric theorem* .
- Proof of the *maximum modulus theorem* in complex analysis.

(2) *Abel summability* and the *Poisson kernel* with applications to the Laplace equation on a disc and the heat equation.

The function space $L^p[a,b]$. We now come to the second mode of convergence namely convergence in mean. However let is first recall some rudiments of Lebesgue theory. It is essential use *Lebesgue intgerals* and we shall work with the *Lebesgue measure* on intervals in the real line. Let I be an interval on the real line. For $1 \leq p < \infty$, we shall denote by $L^p(I)$ the set of all measurable functions on I such that

$$\int_I |f(x)|^p dx < \infty.$$

and on $L^p(I)$ we have the *norm*

$$\|f\|_p = \Big(\int_I |f(t)|^p dt\Big)^{1/p}$$

Remark: With this norm $L^p(I)$ is a *Banach Space* . We also have the space $L^\infty(I)$ but we shall make very little use of it and so we shall not recall it right now. All we need in this chapter is the case $p = 2$ and for bounded intervals mainly $[-\pi, \pi]$. If $g : [a, b] \longrightarrow \mathbb{R}$ is an integrable function, its L^2 norm is

$$\|g\| = \Big(\int_a^b |g(t)|^2 dt\Big)^{1/2}. \tag{2.1}$$

However the integral (2.1) may be $+\infty$. For instance $g(x) = 1/\sqrt{x}$ is integrable on $[0, 1]$ but the integral of $|g(x)|^2$ over $[0, 1]$ is infinite. Thus $/\sqrt{x}$ is integrable on $[0, 1]$ but not square integrable. So we see that $L^1[a, b]$ and $L^2[a, b]$ are distinct. Show that if $-\infty < a < b < \infty$ then

$$L^2[a, b] \subset L^1[a, b].$$

In particular for the interval $[-\pi, \pi]$ every function in $L^2[-\pi, \pi]$ has a Fourier series. We wish to study the convergence of the partial sums with respect to the L^2-norm.

The space $L^2[a, b]$ The space of all functions $g : [a, b] \longrightarrow \mathbb{R}$ for which the integral (2.1) is finite is denoted by $L^2[a, b]$ and it is a vector space. It is evident that if $f \in L^2[a, b]$ then $cf \in L^2[a, b]$ for every scalar c. We need to prove that if f and g are square integrable then $f + g$ is also square integrable. This is easy:

$$\begin{aligned}
|f(x) + g(x)|^2 &\leq (|f(x)| + |g(x)|)^2 \\
&\leq |f(x)|^2 + |g(x)|^2 + 2|f(x)||g(x)| \\
&\leq |f(x)|^2 + |g(x)|^2 + (|f(x)|^2 + |g(x)|^2) \\
&\leq 2(|f(x)|^2 + |g(x)|^2)
\end{aligned}$$

From this it follows at once that $|f+g|$ is in $L^2[a,b]$ completing the verification that $L^2[a,b]$ is a vector space. The vector space $L^2[a,b]$ has a very important property that we now spell out.

Theorem (Cauchy completeness of $L^p[a,b]$): $L^2[a,b]$ is *Cauchy complete* with resect to (2.1).

Let us recall what the last clause means. If (g_n) is a sequence of functions such that

$$\|g_n - g_m\| \longrightarrow 0 \text{ as } m, n \to \infty$$

then there is a $g \in L^2[a,b]$ such that $\|g_n - g\| \longrightarrow 0$ as $n \to \infty$.

We shall not prove the theorem since it is quite standard in analysis courses. See for instance the book of *Goffman and Pedrick - Functional Analysis.*

Trouble with continuous functions! Note that if $g : [a,b] \longrightarrow \mathbb{R}$ is continuous then obviously the integral (2.1) is finite. In other words $C[a,b] \subset L^2[a,b]$.

However there is a sequence (g_n) of continuous functions on $[a,b]$ such that

$$\|g_n - g_m\| \longrightarrow 0 \text{ as } m, n \to \infty$$

but there is NO continuous function g such that

$$\|g_n - g\| \longrightarrow 0 \text{ as } n \to \infty$$

In otherwords $C[a,b]$ is NOT Cauchy complete with respect to the norm $\|g\|$ defined by (2.1). *Cauchy completeness is ESSENTIAL for many important existence results in analysis.*

$L^2[a,b]$ is an innerproduct space On $L^2[a,b]$ we define an innerproduct as follows:

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt \tag{2.2}$$

If we work with complex valued functions we put a bar on the factor $g(t)$ where the bar signifies complex conjugation. Observe that $\|g\|^2 = \langle g, g\rangle$.

Exercises: (i) Prove the parallelogram law: $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$

We say that f and g are *orthogonal* if $\langle f, g\rangle = 0$.

(ii) Prove f and g are orthogonal then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$ (Pythagorous's identity).

(iii) Prove $\|f+g\| \le \|f\| + \|g\|$ (triangle inequality)

(iii) Show that $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$ are pair-wise orthogonal as elements of $L^2[-\pi, \pi]$.

The Legendre polynomials Consider the polynomials $P_n(x)$ given by

$$P_n(x) = \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2-1)^n, \quad n = 0, 1, 2, \ldots \tag{2.3}$$

Check that these polynomials are pairwise orthogonal as elements of $L^2[-1,1]$.

Exercise: (i) Show that $P_n(1) = 1$

(ii) $P_n(x)$ is an odd function if n is odd and an even function if n is even.

(iii) The polynomial $P_n(x)$ has n distinct real roots in $(-1,1)$.

Remark: The roots of the polynomial $P_n(x)$ play an extremely important role in *numerical quadrature* known as *Gaussian quadrature.* See the book of *S. Chandrasekhar, Radiative transfer, Dover Publications, New York, 1960* . For more on Legendre polynomials as well as other orthogonal systems of polynomials, my home page link: *http://www.math.iitb.ac.in/ gopal/MA207/ma207.pdf*

Least square approximation:

We begin with a definition.

definition A *trigonometric polynomial* of degree at most N is a linear combination

$$Q_N(x) = \alpha_0 + \sum_{j=1}^{N}(\alpha_j \cos jx + \beta_j \sin jx)$$

Suppose $f \in L^2[-\pi, \pi]$ then $S_N(f, x)$, the $N-$th partial sum of the Fourier series of $f(x)$, is obviously a trigonometric polynomial of degree at most N. We now compare the approximations

$$\|f(x) - Q_N(x)\| \text{ and } \|f(x) - S_N(f, x)\|$$

The following theorem asserts that $S_N(f, x)$ is the *best* approximation among *all* trigonometric polynomials of degree at most N.

Theorem: Suppose $f \in L^2[-\pi, \pi]$ and $Q_N(x)$ is a trigonometric polynomial of degree at most N. Then

$$\|f(x) - S_N(f, x)\| \leq \|f(x) - Q_N(x)\|.$$

Equality holds if and only if $Q_N(x) = S_N(f, x)$.

Proof: We begin with the trivial equation

$$f(x) - Q_N(x) = (f(x) - S_N(f, x)) + R_N(X)$$

where $R_N(x) = S_N(f, x) - Q_N(x)$ is a trigonometric polynomial of degree at most N.

We shall now show that $(f(x) - S_N(f, x))$ is orthogonal to each of $1, \sin x, \cos x, \ldots, \sin Nx$ and $\cos Nx$. Well, $\cos jx$ is orthogonal to 1, $\sin kx$ and $\cos kx$ when $k \neq j$. Hence,

$$\begin{aligned}
\int_{-\pi}^{\pi}(f(x) - S_N(f, x)) \cos jx dx &= \pi a_j - \int_{-\pi}^{\pi} S_N(f, x) \cos jx dx. \\
&= \pi a_j - \int_{-\pi}^{\pi} a_j \cos^2 jx dx \\
&= \pi a_j - \frac{a_j}{2} \int_{-\pi}^{\pi}(1 + \cos 2jx) dx \\
&= 0.
\end{aligned}$$

Hence $(f(x) - S_N(f, x))$ is orthogonal to $R_N(x)$ and so by Pythagorous's identity we get

$$\|f(x) - Q_N(x)\|^2 = \|f(x) - S_N(x)\|^2 + \|R_N(x)\|^2$$

We see at once that

$$\|f(x) - Q_N(x)\| \geq \|f(x) - S_N(x)\|$$

and equality holds if and only if $R_N(x) = 0$ that is if and only if $Q_N(x) = S_N(f, x)$.

Theorem (Bessel's inequality): If a_0, a_n, b_n $(n = 1, 2, \dots)$ are the Fourier coefficients of a function $f \in L^2[-\pi, \pi]$ we have the following inequality:

$$|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx \tag{2.4}$$

Proof: We have already seen that $1, \cos x, \sin x, \dots, \cos Nx, \sin Nx$ are orthogonal to $f(x) - S_N(f, x)$ whereby we conclude their linear combination $S_N(f, x)$ is also orthogonal to $f(x) - S_N(f, x)$. The Pythagorous identity now gives:

$$\|f(x) - S_N(f, x)\|^2 + \|S_N(f, x)\|^2 = \|f\|^2 \tag{2.5}$$

Hence

$$\|S_N(f, x)\|^2 \leq \|f\|^2 \tag{2.6}$$