Fourier Analysis and its Applications

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07 Bessel's function of the first kind

The Bessel's functions $J_k(z)$ are of course defined for all k not just for integer values of k and they satisfy the ordinary differential equation (1.47). However we shall use $J_k(z)$ only for *integer values of* k . Further for *convenience* if $k < 0$ and is an *integer* we set

$$J_k(z) = (-1)^k J_{-k}(z) \tag{1.50}$$

Definition: The *generating function* for the sequence of Bessel functions $J_k(z)$ ($k \in \mathbb{Z}$) is defined as

$$G(z,t) = \sum_{k=-\infty}^{\infty} t^k J_k(z) \tag{1.51}$$

We must first discuss the convergence properties of the series (1.51). For this we need to obtain an elementary estimate for $|J_k(z)|$: From the definition

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!}\Big(\frac{z}{2}\Big)^{k+2n}$$

taking absolute values,

$$|J_k(z)| \leq \sum_{n=0}^{\infty} \frac{1}{n!(n+k)!}\Big(\frac{|z|}{2}\Big)^{k+2n}$$

Let us multiply and divide by $(2n+k)!$

$$|J_k(z)| \leq \sum_{n=0}^{\infty} \frac{(2n+k)!}{n!(n+k)!}\frac{1}{(2n+k)!}\Big(\frac{|z|}{2}\Big)^{k+2n}$$

Now we recall that estimate for the binomial coefficient:

$$\binom{2n+k}{n} = (2n+k)!/(n!(n+k)!) \leq 2^{2n+k}$$

Using this we get

$$\begin{aligned} |J_k(z)| &\leq \sum_{n=0}^{\infty} \frac{|z|^{k+2n}}{(2n+k)!} \\ &\leq \frac{|z|^k}{k!}\sum_{n=0}^{\infty} \frac{k!|z|^{2n}}{(2n+k)!} \\ &\leq \frac{|z|^k}{k!}\sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} \end{aligned}$$

So finally the useful estimate

Lemma: $|J_k(z)| \leq \dfrac{|z|^k}{k!}\cosh|z|$. We now use this lemma to derive an important formula due to Schlömilch. As an Exercise Show that the series (1.29) for $G(z,t)$ converges for all $(z,t) \in \mathbb{C}\times(\mathbb{C}-\{0\})$ and that the convergence is uniform on compact subsets. Since the terms are analytic functions of z, term by term differentiation is valid.

Theorem

$$\sum_{k=-\infty}^{\infty} t^k J_k(z) = \exp\left(\frac{zt}{2} - \frac{z}{2t}\right) \tag{1.52}$$

Denoting the sum on the left by $G(z,t)$ we differentiate term by term and obtain

$$\frac{\partial G}{\partial z} = \sum_{k=-\infty}^{\infty} t^k J_k'(z) = \frac{1}{2}\sum_{k=-\infty}^{\infty} t^k (J_{k-1}(z) - J_{k+1}(z))$$

since $J_{k-1}(z) - J_{k+1}(z) = 2J_k'(z)$. Hence

$$\frac{\partial G}{\partial z} = \frac{t}{2}\sum_{k=-\infty}^{\infty} t^{k-1} J_{k-1}(z) - \frac{1}{2t}\sum_{k=-\infty}^{\infty} t^{k+1} J_{k+1} = \Big(\frac{t}{2} - \frac{1}{2t}\Big) G(z,t)$$

From this we immediately get

$$G(z,t) = G(0,t)\exp\left(\frac{tz}{2} - \frac{z}{2t}\right)$$

Observe that $J_k(0) = 0$ if $k > 1$ and $J_0(0) = 1$ from which we conclude $G(0,t) = 1$ for all $t \neq 0$. Thereby,

$$G(z,t) = \exp\left(\frac{tz}{2} - \frac{z}{2t}\right)$$

and the proof of *Schlömilch's formula* is complete.

Integral representation of Bessel functions: We now derive an integral expression for $J_k(z)$ when k is a non-negative integer.

Theorem

$$J_k(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - k\theta)d\theta, \quad k \in \mathbb{Z}, k \geq 0. \tag{1.53}$$

Let us put $t = \exp(i\theta)$ in Schlömilch's formula (1.52) and we get

$$\sum_{k=-\infty}^{\infty} J_k(z)e^{ik\theta} = \exp z\Big(\frac{1}{2}(e^{i\theta} - e^{-i\theta})\Big) = \exp(iz\sin\theta) \tag{1.54}$$

which is a Fourier series for the generating function. Multiply (1.54) by $e^{-ik\theta}$ and integrate over $[-\pi,\pi]$ and we get

$$J_k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(iz\sin\theta - ik\theta)d\theta$$

Writing

$$\exp(iz\sin\theta - ik\theta) = \cos(z\sin\theta - k\theta) + i\sin(z\sin\theta - k\theta)$$

we get the desired result. Note that the integral involving $\sin(z\sin\theta - k\theta)$ vanishes since it is an odd function.

We shall later see an application of this integral representation in a problem on celestial mechanics namely the inversion of the Kepler equation mentioned earlier.

Exercises: (i) Check directly by differentiating the integral (1.53) that $J_k(z)$ satisfies the differential equation

$$z^2y'' + zy' + (z^2 - k^2)y = 0.$$

(ii) Show that $xJ_0(x)$ is a solution of $y'' + y = -J_1(x)$.

(iii) Show that

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y).$$

Use of the integral representation We would like to emphasize that the integral representation

$$J_k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z\sin\theta - k\theta)d\theta$$

can be used for the following purposes:

(i) To show that $J_k(z)$ has infinitely many real zeros

(ii) To obtain decay properties of $J_k(x)$ as $x \longrightarrow \infty$ along the real axis.

(iii) To obtain asymptotic behaviour of $J_k(x)$ for x large and real.

We end this chapter with a few exercises from the book of *T. W. Körner, Exercises for Fourier Analysis, CUP, 1993.*

1. Consider the function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ given by

$$f(x) = \frac{1}{12}(3x^2 - \pi^2), \quad x \in [-\pi, \pi].$$

Show that

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

2. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin nx$$

3. Show that the following function $f(x, y)$ is continuous on $\mathbb{R}^2$ and find its zeros.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\sin nx \sin ny$$

Further results relegated for later time We must now conclude this chapter here. We shall take up some of the following items later (starred items will be taken up only if time permits)

1. Proof of Dirichlet's theorem concerning piecewise monotone functions
2. Existence of continuous functions whose Fourier series diverges at specified points
3. $*$ Hardy's proof of the functional equation for the Riemann zeta function.
4. $*$ Fourier expansions for Bernoulli functions.

END of CHAPTER - 1