Fourier Analysis and its Applications

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60 The Poisson summation formula

Continuity of differentiation We have seen that for a sequence $\{f_n\}$ in $\mathcal{S}(\mathbb{R})$, convergence in $\mathcal{S}(\mathbb{R})$ is a *VERY STRONG* notion and we know that if $\{f_n\}$ converges to f in $\mathcal{S}(\mathbb{R})$ then the sequence of derivatives $\{f_n'\}$ converges to f' in $\mathcal{S}(\mathbb{R})$. In other words differentiation is a continuous operator on $\mathcal{S}(\mathbb{R})$.

We now show that differentiation of distributions is a *sequentially* continuous operaror on $\mathcal{S}'(\mathbb{R})$

Theorem (Continuity of differentiation): Suppose $\{u_\nu\}$ is a sequence of tempered distributions *converging weakly* to u then the sequence $\{u_\nu'\}$ converges *weakly* to u'.

Proof: Well, let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary. Then

$$\langle u_n', g\rangle = -\langle u_n, g'\rangle \longrightarrow -\langle u, g'\rangle = \langle u', g\rangle.$$

This proves the theorem.

Recall that the Fourier transform was continuous as an operator on $\mathcal{S}(\mathbb{R})$. We now establish the *sequential continuity* of the Fourier transform as an operator on the space $\mathcal{S}'(\mathbb{R})$.

Theorem (Continuity of the Fourier transform): Suppose $\{u_\nu\}$ is a sequence of tempered distributions *converging weakly* to u then the sequence $\{\widehat{u_\nu}\}$ converges *weakly* to $\widehat{u}$.

Proof: Well, let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary. Then

$$\langle \widehat{u_n}, g\rangle = \langle u_n, \widehat{g}\rangle \longrightarrow \langle u, \widehat{g}\rangle = \langle \widehat{u}, g\rangle.$$

This proves the theorem.
Exercise:

28. Find the weak limit
$$\lim_{\epsilon\to 0+} \frac{\sqrt{\pi}}{\sqrt{\epsilon}} \exp(-x^2/4\epsilon).$$

29. Consider $u = \exp(iax^2)$ where a is a real number. This is a tempered distribution. Establish the following the weak limit
$$\lim_{\epsilon\to 0+} \exp(i(a+i\epsilon)x^2) = u$$
Use the dominated convergence theorem. Now for $\epsilon > 0$, we have that $x \mapsto \exp(i(a+i\epsilon)x^2)$ is a function in $\mathcal{S}(\mathbb{R})$ and so its Fourier transform can be computed via the usual formula using integrals. Well, we encounter the following
$$\exp(-\xi^2/4(\epsilon+ia)) \int_{\mathbb{R}} \exp(-\lambda(x+\frac{i\xi}{2\lambda})^2)dx, \quad \lambda = \epsilon + ia.$$
As in chapter 1, use Cauchy's integral theorem to shift the contour of integration. Finally let $\epsilon \to 0+$ and use the continuity of the Fourier transform as an operator on $\mathcal{S}'(\mathbb{R})$. For a slightly different approach see *Strichartz, p. 47*.

Some examples from Strichartz' book (page 50) Here we discuss some very nice examples from the book of *R. Strichartz* cited earlier. However we shall take a slightly different approach. These are distributional analogues of radial Fourier transforms discussed earlier. Recall that a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is said to be radial if

$$\phi \circ R = \phi \tag{10.20}$$

for all $R \in \mathrm{SO}(n, \mathbb{R})$. There is a way to formulate an analogue of (10.20) for distributions that go under the name of (*compositions with smooth maps*). The rule is not difficult to motivate and describe (see chapter 6 of *L. Hörmander's* book). Since in the examples we look at only distributions that are represented by functions, the usual composition rule (10.20) can be taken as a working definition. Here we look at the examples of $|x|^{-1}$ and $|x|^{-2}$ as tempered distributions in $\mathcal{S}'(\mathbb{R}^3)$

Theorem: The locally integrable functions $|x|^{-1}$ and $4\pi|x|^{-2}$ define tempered distributions on $\mathbb{R}^3$ via the prescriptions

$$g \mapsto \int_{\mathbb{R}^3} \frac{g(x)dx}{|x|^{3-k}}, \quad k = 1, 2. \tag{10.21}$$

Further the Fourier transform of $|x|^{-1}$ is the distribution $4\pi|x|^{-2}$.

It follows at once from the dominated convergence theorem that the prescriptions (10.21) do define tempered distributions. The Fourier transform computation is of course a very different matter. Call the distribution $|x|^{-1}$ as u and proceed naively:

$$\langle \widehat{u}, g\rangle = \langle u, \widehat{g}\rangle.$$

Now appealing to (10.21) the RHS equals

$$\int_{\mathbb{R}^3} \frac{\widehat{g}(x)dx}{|x|} = \int_{\mathbb{R}^3} \frac{dx}{|x|} \int_{\mathbb{R}^3} \exp(-ix\cdot y)g(y)dy$$

Switching the order of integrals would land us in trouble. The RHS gives us

$$\int_{\mathbb{R}^3} g(y)dy \int_{\mathbb{R}^3} \frac{\exp(-ix\cdot y)dx}{|x|}$$

Put $x = Az$ where A is an orthogonal matrix and $A^T y = |y|\mathbf{e}_3$ so that the integral (in polar coordinates for the inner one) takes the form

$$\int_{\mathbb{R}^3} g(y)dy \int_{\mathbb{R}^3} \frac{\exp(-iz\cdot|y|\mathbf{e}_3)dz}{|z|} = 2\pi \int_{\mathbb{R}^3} g(y)dy \int_0^\infty \rho d\rho \int_0^\pi \exp(-i\rho|y|\cos\phi)\sin\phi d\phi.$$

Put $\cos\phi = t$ and perform one integration. We get

$$\langle \widehat{u}, g\rangle = 4\pi \int_{\mathbb{R}^3} \frac{g(y)}{|y|}dy \int_0^\infty \sin(|y|\rho)d\rho$$

We have to deal with the oscillatory integral $\int_0^\infty \sin\rho|y|d\rho$. One can of course resort to the usual $\exp(-\epsilon|x|^2)$ trick. Try it out and you would find it slightly *troublesome*. A slightly easier method is to modify the trick and instead introduce $\exp(-\epsilon|x|)$ instead. You will then get that

$$\langle \widehat{u}, g\rangle = \lim_{\epsilon\to 0+} 4\pi \int_{\mathbb{R}^3} \frac{g(y)dy}{|y|}dy \int_0^\infty e^{-\epsilon\rho}\sin(|y|\rho)d\rho$$

If you remember the formula for the *Laplace transform of* $\sin \omega t$ then we can save some time and write

$$\langle \widehat{u}, g\rangle = \lim_{\epsilon\to 0+} 4\pi \int_{\mathbb{R}^3} \frac{g(y)dy}{|y|^2+\epsilon^2} = \langle \frac{4\pi}{|y|^2}, g\rangle$$

as desired. Exercise: Check the last equality (*weak convergence in distributions*).

Distributional convergence of Fourier series Let us go back to the example from chapter 1 on the Fourier expansion of $|x|$:

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4\cos(2k-1)x}{\pi(2k-1)^2}$$

The Fourier series converges uniformly to the periodic extension of $|x|$ and so in particular the partials sums $S_n(f,x)$ converge in the sense of distributions as well. Now the continuity of differentiation gives the Fourier expansion for the signum function:

$$\mathrm{sgn}(x) = 4\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{\pi(2k-1)}$$

In the last chapter the *classical Dirichlet's theorem* gave us pointwise convergence whereas here we get the *distributional convergence*. Let us differentiate again which is clearly permissible in the context of *distributional* convergence. The distributional derivative of the signum function is $2\delta_0$ (how?) and remember that we have the periodic extension of the signum function which means *there will appear a Dirac delta concentrated at each of the points* $\pm\pi, \pm 2\pi, \ldots$ whereby we get the Fourier expansion of the *periodized version Dirac distribution*:

$$\frac{\pi}{2}\sum_{n=-\infty}^{\infty} (-1)^n \delta_{\pi n} = \sum_{k=1}^{\infty} \cos(2k-1)x \tag{10.22}$$

It would be troublesome to deal with the series on the RHS along classical lines!

29. Explain the reason for the appearence of $(-1)^n$ in the last equation.

30. Take the Fourier transform on both sides of the last equation and compare it with equation (1.25) in chapter 1. Equation (10.22) is the Jacobi theta function identity in disguise !

The theta function revisited Let us look at the distributional equation (10.22) closely. Take $g(x) = \exp(-tx^2)$ which is in the Schwartz class ($t > 0$). Applying both sides of (10.22) to this $g(x)$ we get

$$\frac{\pi}{2}\sum_{n=-\infty}^{\infty} (-1)^n e^{-t\pi^2 n^2} = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{-tx^2} \cos((2k-1)x)dx.$$

This can be re-written as

$$\pi\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi^2 n^2 t}\Big) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} e^{-tx^2}(e^{ix(2k-1)} + e^{-ix(2k-1)})dx.$$

The RHS is of course sum of Fourier transforms of the Gaussian and we get

$$\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\pi^2 n^2 t}\Big) = 2\sum_{k=1}^{\infty}\sqrt{\frac{1}{\pi t}}\exp(-(2k-1)^2/4t).$$

Replace t by $1/(4\pi^2\tau)$ and we get

$$\frac{1}{2\sqrt{\pi\tau}}\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2/(4\tau)}\Big) = \sum_{k=-\infty}^{\infty}\exp(-\pi^2(2k-1)^2\tau). \tag{10.23}$$

Let us now recall the formula for the Jacobi theta function obtained in chapter 1:

$$\sum_{n=-\infty}^{\infty}\exp(-\tau(t+2\pi n)^2) = \frac{1}{2\sqrt{\tau\pi}}\Big(1 + 2\sum_{n=1}^{\infty}\exp(-n^2/4\tau)\cos nt\Big) \tag{1.25}$$

In perfect accordance when we set $t = \pi$ in (1.25). We now look at a profound generalization of these classical results.

The Poisson summation formula The LHS of equation (10.22) is a difference of two series of the form

$$u = \sum_{m\in\mathbb{Z}}\delta_{mc}, \quad c \neq 0. \tag{10.24}$$

Note that the partial sums of (10.24) converges weakly (Exercise) and so the sum is a tempered distribution. We now study this distribution closely omitting routine computational details. The partial sums of (10.24) converges *weakly in sense of distributions:*

$$\widehat{u} = \sum_{m\in\mathbb{Z}} e^{-iymc} = 1 + \sum_{m=1}^{\infty} 2\cos mcy = 1 + F''(y) \tag{10.25}$$

where F is the $2\pi/c$ periodic even function given by:

$$F(y) = -\frac{2}{c^2}\sum_{m=1}^{\infty}\frac{\cos mcy}{m^2} \tag{10.26}$$

To identify F, look at the *saw-tooth* function $s(x)$ given as $s(x) = x$ on $(0, 2\pi)$ extended with period 2π. Determine the Fourier series for $s(x)$ and integrate it term by term. We get

$$\int_0^x s(t)dt = \frac{\pi^2}{3} - 2\sum_{m=1}^{\infty}\frac{\cos mx}{m^2}, \quad x \in [0, \pi].$$

Extend the above as an even function to $[-\pi, \pi]$ and then deduce that F is given by

$$F(y) = -\frac{\pi^2}{3c^2} + \frac{\pi|y|}{c} - \frac{y^2}{2}, \quad -2\pi/c \le y \le 2\pi/c.$$

It is clear that the second distributional derivative of F is

$$F'' = -1 + \frac{2\pi}{c}\sum_{m\in\mathbb{Z}}\delta_{\frac{2\pi m}{c}} \tag{10.27}$$

Substituting into (10.25) gives us the remarkable result known as the *Poisson summation formula*:

Theorem For $c \neq 0$, let $u = \sum_{m\in\mathbb{Z}} \delta_{mc}$ then

$$\widehat{u} = \frac{2\pi}{c} \sum_{m\in\mathbb{Z}} \delta_{2\pi m/c} \tag{10.28}$$

Remarks: The distribution u above is a lattice of point masses - one placed at each lattice point mc. The Fourier transform is then another lattice of Dirac deltas ! Applied to a function $\phi \in \mathcal{S}(\mathbb{R})$ equation (10.28) assumes the classical aspect (*In chapter 1, the $\phi(x)$ was the Gaussian*):

$$\sum_{m\in\mathbb{Z}} \phi(mc) = \frac{2\pi}{c} \sum_{m\in\mathbb{Z}} \widehat{\phi}(2\pi m/c). \tag{10.29}$$

Poisson summation formula. Concluding remarks. The road ahead. The Poisson summation formula assumes a more interesting aspect in several variables. The book of *R. Strichartz (p. 120)* contains a discussion of the notions of *lattices* and *dual lattices*. If we think of a lattice of Dirac deltas as a crystal structure then the Fourier transform corresponds to X-ray diffraction patterns showing bright spots at the points of the dual lattice. The student can scarcely do better than to undertake a systematic study of *Strichartz's book* for matters such as

1. Bochner's theorem on positive definite functions
2. Heisenberg's uncertaintly principle
3. Paley-Wiener theorems and a glimpse into micro-local analysis
4. A glimpse into Wavelets and the remarkable Haar systems
5. A glimpse into pseudo-differential operators

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