Fourier Analysis and its Applications

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06 The Riemann zeta function

Analytic continuation of $\zeta(s)$ on $\mathbb{C} - \{1\}$.

1. Start with the simple formula:
$$\int_0^\infty e^{-nt} t^{s-1} dt = \Gamma(s) n^{-s} \tag{1.43}$$
Set $n = 1, 2, 3, \ldots$ and add, justifying carefully the exchange of infinite sums and integrals.

2. Establish the formula
$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1} dt}{e^t - 1} = \int_0^\infty t^{s-2} \phi(t) dt, \quad \text{Re } s > 1. \tag{1.44}$$

3. Study the function $\phi(t) = t/(e^t - 1)$ for $t \neq 0$ and show that with $\phi(0) = 1$ the function is infinitely differentiable on the real line.

4. Perform and integration by parts and show that for Re $s > 1$ the following holds:
$$\zeta(s) = \frac{-1}{(s-1)\Gamma(s)} \int_0^\infty t^{s-1} \phi'(t) dt \tag{1.45}$$
However the integral on the right hand side of (1.45) is holomorphic on Re $s > 0$ and so we get an analytic continuation of the zeta function on the larger domain Re $s > 0$ and $s \neq 1$.

5. Show that further integration by parts gives
$$\zeta(s) = \frac{(-1)^{k-1}}{(s-1)\Gamma(s+k)} \int_0^\infty t^{s+k-1} \Big(\frac{d}{dt}\Big)^{k+1} \Big(\frac{t}{2} \coth\Big(\frac{t}{2}\Big)\Big) dt, \quad \text{Re } s > -k. \tag{1.46}$$
Deduce that $\zeta(s)$ is holomorphic on $\mathbb{C} - \{1\}$.

Fourier series and Bessel's functions

We now turn to another important special function namely the *Bessel function* of the first kind which appears commonly in problems concerning wave propagation. We list some important reasons for studying these:

(1) Vibrations of a circular membrane. Standing waves.

(2) Problems involving cylinderical symmetries. *Newton's rings.*

(3) Problems in analytic number theory.

(4) A problem in celestial mechanics. Inversion of the *Kepler equation.*

For (3) the book *H. Iwaniec and E. Kowalski, Analytic Number Theory, American Math. Soc, Providence, RI, 2004.* is highly recommended.

The Bessel's differential equation: This is the ordinary differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad p \geq 0. \tag{1.47}$$

Writing this as

$$x^2y'' + xy' - p^2y = -x^2y,$$

we can think of (1.47) as a perturbation of the Cauchy Euler equation

$$x^2y'' + xy' - p^2y = 0. \tag{1.48}$$

Since (1.48) has solution x^p we are motivated to look for a solution of (1.47) as a Frobenius series

$$y(x) = x^p(a_0 + a_1x + a_2x^2 + \dots), \quad a_0 \neq 0.$$

Computation of the a_n is routine and will not be done here. With suitable normalization...

Bessel's functions of the first kind Recall the definition of Bessel's functions of the first kind:

Definition: The *Bessel's function of the first kind* of order k (where $k \in \mathbb{Z}$ and $k \geq 0$) is denoted by $J_k(z)$ and defined to be the sum of the series:

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{z}{2}\right)^{k+2n} \tag{1.49}$$

Exercises: (i) Show that the series converges for all $z \in \mathbb{C}$ and defines an entire function. Term by term differentiation is valid.

(ii) Prove $(z^k J_k(z))' = z^k J_{k-1}(z)$ and $(z^{-k} J_k(z))' = -z^{-k} J_{k+1}(z)$

(iii) Prove $J_{k-1}(z) - J_{k+1}(z) = 2J_k'(z)$ and $J_{k+1}(z) + J_{k-1}(z) = 2kz^{-1}J_k(z)$ The Bessel's functions $J_k(z)$ are of course defined for all k not just for integer values of k and they satisfy the ordinary differential equation (1.47). However we shall use $J_k(z)$ only for *integer values of* k. Further for *convenience* if $k < 0$ and is an *integer* we set

$$J_k(z) = (-1)^k J_{-k}(z) \tag{1.50}$$

Definition: The *generating function* for the sequence of Bessel functions $J_k(z)$ ($k \in \mathbb{Z}$) is defined as

$$G(z,t) = \sum_{k=-\infty}^{\infty} t^k J_k(z) \tag{1.51}$$

We must first discuss the convergence properties of the series (1.51). For this we need....