Fourier Analysis and its Applications
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59 Distributional solutions of ODEs. Continuity of differentiation and the Fourier transform

Now that we have a notion of differentiation of a distribution and the operation of multiplication by polynomials we can ask whether the classical differential equations

$$\begin{aligned}(1-x^2)y'' - 2xy' + p(p+1)y &= 0\\ (1-x^2)y'' - xy' + p^2y &= 0\\ x^2y'' + xy' + (x^2-p^2)y &= 0\\ xy'' + (1-x)y' + \lambda y &= 0\\ x(1-x)y'' + (c-(a+b+1)x)y' - aby &= 0\end{aligned}$$

have solutions in spaces of distributions. The classical theory of ODEs such as the vector space of solutions being equal to the order of the differential equation FAILS across singular points namely the points ± 1 in the first two examples and the origin in the last three examples. In the last one both 1 and 0 are singular points.

The case of the Cauchy Euler equation Rather than discuss these things in general let us look at one specific example and show that for a second order Cauchy Euler equation, the space of solutions can be at least *four dimensional*. Consider

$$x^2y'' + xy' - y = 0 \tag{10.18}$$

We know that on $(0,\infty)$ there are two linearly independent solutions x and $1/x$. The solution x is defined on the entire real line. Zero function is also a solution. Now let us consider the solution

$$y_1(x) = xH(x) \tag{10.19}$$

which is a tempered distribution. Exercises:

24. Check that $y_1' = H(x)$ and $y_1'' = \delta_0$. Substituting into (10.18) we see at once that $y_1(x)$ satisfies the ODE.

25. Check that δ_0 is also a solution of the ODE.

26. Check that $\mathrm{PV}(\frac{1}{x})$ satisfies the ODE. Well, abbreviating the distribution by y we have to check that for each $g \in \mathcal{S}(\mathbb{R})$
$$\langle x^2y'' + xy' - y, g\rangle = 0.$$
This is equivalent to checking
$$\langle y, (x^2g)'' - (xg)' - g\rangle = 0.$$
The left hand side is
$$\lim_{\epsilon\to 0+}\int_{|x|\geq\epsilon}(g' + xg'')dx = \int_{\mathbb{R}}(g' + xg'')dx$$
which is easily seen to be zero.

27. Show that the four solutions $x, xH(x), \delta_0$ and $\mathrm{PV}(\frac{1}{x})$ are linearly independent. Is every solution of the ODE a linear combination of these?

We shall say no more regarding these things. The relevant material from distribution theory needed for a systematic handling of such matters may be found in
L. Hörmander, Analysis of linear partial differential operators - I, Springer Verlag, 1990, p 68 ff.

Convergence of distributions Since the space of distributions is the dual space to the locally convex TVS $\mathcal{S}(\mathbb{R})$, it carries the *weak star topology.* We define the notion of weak (star) sequential convergence

Definition: A sequence $\{u_\nu\}$ of tempered distributions is said to converge to u *weakly* if for each $g \in \mathcal{S}(\mathbb{R})$ we have

$$\langle u_\nu, g\rangle \longrightarrow \langle u, g\rangle, \quad \nu \to \infty.$$

This is a VERY weak form of convergence comparable to weakly convergent sequence in a Hilbert space that we discussed in chapter 7. Here is our example revisited:

(1) $\sin \nu x$ is a tempered distribution and this converges weakly to zero as $\nu \to \infty$.

(2) Show that $\exp(-x^2/n)$ converges weakly to the constant 1 as $n \to \infty$.

(3) Does $\nu \sin \nu x$ converge weakly to zero as $\nu \to \infty$? What about $\nu^k \sin \nu x$?

If a sequence $\{u_\nu\}$ in $L^2(\mathbb{R})$ converges weakly in $L^2(\mathbb{R})$ then obviously the sequence regarded as tempered distributions converges weakly. Is the converse true? No. This further suggesting that weak convergence of distributions is *REALLY VERY WEAK* notion. However it is a useful notion in analysis since many existence theorems are proved by first capturing the desired object as some form of weak limit and then using other techniques to establish the desired regularity of the (weak) limit. *With such a weak notion of convergence there is a better chance of hitting the limit* ! Also many Fourier series we encounter may be divergent and we need tools to handle divergent series - for instance Cesaro summability or Abel summability. Weak (distributional) convergence of a sequence gives *another tool* for handling *divergent* series.
There other places where you may have encountered this type of convergence namely in measure theory and probability. This is weak convergence of measures. Recall that a regular Borel measure with compact support on the real line is a tempered distribution.

Continuity of differentiation We have seen that for a sequence $\{f_n\}$ in $\mathcal{S}(\mathbb{R})$, convergence in $\mathcal{S}(\mathbb{R})$ is a *VERY STRONG* notion and we know that if $\{f_n\}$ converges to f in $\mathcal{S}(\mathbb{R})$ then the sequence of derivatives $\{f_n'\}$ converges to f' in $\mathcal{S}(\mathbb{R})$. In other words differentiation is a continuous operator on $\mathcal{S}(\mathbb{R})$.

We now show that differentiation of distributions is a *sequentially* continuous operaror on $\mathcal{S}'(\mathbb{R})$

Theorem (Continuity of differentiation): Suppose $\{u_\nu\}$ is a sequence of tempered distributions *converging weakly* to u then the sequence $\{u_\nu'\}$ converges *weakly* to u'.

Proof: Well, let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary. Then

$$\langle u_n', g\rangle = -\langle u_n, g'\rangle \longrightarrow -\langle u, g'\rangle = \langle u', g\rangle.$$

This proves the theorem.

Recall that the Fourier transform was continuous as an operator on $\mathcal{S}(\mathbb{R})$. We now establish the *sequential continuity* of the Fourier transform as an operator on the space $\mathcal{S}'(\mathbb{R})$.

Theorem (Continuity of the Fourier transform): Suppose $\{u_\nu\}$ is a sequence of tempered distributions *converging weakly* to u then the sequence $\{\widehat{u_\nu}\}$ converges *weakly* to $\widehat{u}$.

Proof: Well, let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary. Then

$$\langle \widehat{u_n}, g \rangle = \langle u_n, \widehat{g} \rangle \longrightarrow \langle u, \widehat{g} \rangle = \langle \widehat{u}, g \rangle.$$

This proves the theorem.