Fourier Analysis and its Applications

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56 Operations on distributions

One can differentiate a distribution and the following theorem clarifies this idea. Recall that if $f \in \mathcal{S}(\mathbb{R})$ then f' also lies in $\mathcal{S}(\mathbb{R})$ and the map $f \mapsto f'$ is continuous.

Theorem (Differentiating a tempered distribution): Suppose $u \in \mathcal{S}'(\mathbb{R})$ then define a map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ via the prescription:

$$f \mapsto -u(f') \tag{10.6}$$

Then (10.6) defines a distribution denoted by u' known as *the derivative of* u (in the sense of distributions). Further if u is also continuously differentiable in the classical sense and vanishes outside a compact set then the derivative of u in the sense of distributions agrees with the classical derivative of u.

Let us first clarify the meaning of the last clause in the theorem. Denote by u' the classical derivative of u and Du the distributional derivative.

Comparing classical derivative and distributional derivative Assume u is of class C^1 and vanishes outside $[-R, R]$ then (10.6) reads:

$$(Du)(f) = -u(f') = -\int_{-R}^{R} u(x)f'(x)dx = \int_{\mathbb{R}} u'(x)f(x)dx = u'(f).$$

The RHS above is the distribution defined by the $L^p(\mathbb{R})$ function u' (see example 3 above given by (10.5)). In other words as functions Du and u' agree as functions $\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$. Thus the *notion of distributional derivative generalizes the notion of derivative in the sense of elementary calculus.*
Example: Consider the Heaviside function H which takes the value 1 on $(0, \infty)$ and value 0 on $(-\infty, 0)$. Being in $L^\infty(\mathbb{R})$, it is a tempered distribution. Let us compute its distributional derivative. Well, we can now denote the (distributional) derivative as H' and for $f \in \mathcal{S}(\mathbb{R})$,

$$H'(f) = -H(f') = -\int_0^\infty H(x)f'(x)dx = -\int_0^\infty f'(x)dx = f(0) = \delta_0(f)$$

Thus, $H' = \delta_0$.

Differentiation and multiplication by polynomials. Examples contd...

8. Show that $x\delta_0 = 0$ and $x\delta_0' = -\delta_0$. Well, call the second one u and

$$u(f) = \delta_0'(xf) = -\delta_0((xf)') = -(xf)'(0) = -f(0) = -\delta_0(f).$$

9. Show that $x\mathrm{PV}(\frac{1}{x}) = 1$.

10. Show that $\ln|x|$ is a tempered distribution and find its first two derivatives.

Obviously a distribution can be differentiated repeatedly!

Remarks: The derivative of the Dirac delta has a physical interpretation. It is related to the notion of an *electric dipole* and the second derivative is related to the *electric quadrupole*. See the discussion (on page 35) in the book of *I. Richards and H. Youn* cited earlier. In higher dimensions the matter gets more interesting. See the discussion of multipole expansions in the book of *L. Sirovich* cited earlier.

Pairing: It is customary in Functional analysis to denote the pairing $u(x)$ of a vector $x \in V$ and an element $u \in V^*$ by $\langle u, x \rangle$. In other words

$$u(x) = \langle u, x \rangle.$$

This is particularly useful in the context of distributions since a tempered distribution is an element of the dual space of $\mathcal{S}(\mathbb{R})$ and we shall use this notation frequently. In the simplest examples where an element of $u \in L^p(\mathbb{R})$ is regarded as a distribution this pairing assumes the form of an integral:

$$\langle u, f \rangle = \int_{\mathbb{R}} u(x) f(x) dx. \tag{10.7}$$

This suggests that distributional pairing is some kind of generalization of an integral. Indeed a regular Borel measure is an element of the dual space of $C_c(X)$ where X is a locally compact Hausdorff space (for example an open set in $\mathbb{R}^n$).

Support of a distribution: Recall that for a continuous complex valued function $u : \mathbb{R}^n \longrightarrow \mathbb{C}$, the support of u is defined as the closure of the set of points

$$\{x \in \mathbb{R}^n \ : \ u(x) \neq 0\}.$$

The support of u is denoted by $\text{supp}(u)$. Loosely speaking it is the smallest closed set outside which the function is identically zero. Let us recast it differently.
Exercise: Show that $p \notin \text{supp}(u)$ if and only if there exists a neighborhood N_p of p such that

$$\int_{\mathbb{R}^n} u(x) f(x) dx = 0, \qquad \text{for all } f \in C_c^\infty(N_p).$$

In other words,

$$\langle u, f \rangle = 0, \text{ for all } f \in C_c^\infty(N).$$

The advantage of this re-formulation is that the definition carries over to distributions!

Definition (Support of a distribution): Let u be a tempered distribution. Then the *support of* u denoted by $\text{supp}(u)$ is the complement of the set of points p having a neighborhood N_p such that

$$\langle u, f \rangle = 0, \text{ for all } f \in C_c^\infty(N_p).$$

Evidently the complement of the support is an open set.

1. The Heaviside function has support $[0, \infty)$.

2. The Dirac distribution δ_0 has support $\{0\}$.

3. It is clear that $\text{supp}(u') \subset \text{supp}(u)$. The inclusion can be strict as can be seen taking $u = H$.

4. The zero function has support as the empty set.

If the distribution u has compact support we say that u is a compactly supported distribution. These play an important role in the theory.

We now state an important theorem without proof.

Theorem (Distribution with point support): Suppose u is a distribution with support at the point $\{0\}$ then there exists constants $c_0, c_1, \ldots, c_n$ such that

$$u = \sum_{j=0}^{n} c_j \delta_0^{(j)}$$

where $\delta_0^{(j)}$ is the j-th derivative of δ_0 namely the distribution

$$f \mapsto (-1)^j f^{(j)}(0), \quad f \in \mathcal{S}(\mathbb{R}).$$

For completeness we also mention here the Dirac distribution at a general point $p \in \mathbb{R}$ which is the distribution given by

$$f \mapsto f(p).$$

Formulate analogue of the theorem for distributions with point support $\{p\}$.

Fourier transform of a tempered distribution Before we take this up let us prove a simple lemma for the case of two functions in $\mathcal{S}(\mathbb{R})$

Theorem: Suppose $f, g \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle \tag{10.8}$$

Proof: Well,

$$\begin{aligned} \text{LHS} = \int_{\mathbb{R}} \widehat{f}(x) g(x) dx &= \int_{\mathbb{R}} g(x) dx \int_{\mathbb{R}} f(y) \exp(-ixy) dy \\ &= \int_{\mathbb{R}} f(y) dy \int_{\mathbb{R}} g(x) \exp(-ixy) dx \\ &= \int_{\mathbb{R}} f(y) \widehat{g}(y) dy = \text{RHS} \end{aligned}$$

Equation (10.8) suggests a definition of the Fourier transform of a tempered distribution.