Fourier Analysis and its Applications

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55 Examples of tempered distributions

Theorem: The Fourier transform $f \mapsto \widehat{f}$ as a linear operator on $\mathcal{S}(\mathbb{R})$ is continuous with respect to the topology of $\mathcal{S}(\mathbb{R})$.

Recall that in chapter 4 we extended the Fourier transform from $\mathcal{S}(\mathbb{R})$ to an operator on $L^2(\mathbb{R})$. Now is there a way to extend it any further ? For example can we extend it to all $L^p(\mathbb{R})$ and even beyond. We shall now see that this is possible. In fact one can even define the Fourier transform of functions like $\exp(ita)$ $(a \in \mathbb{R})$ which don't have any decay and even define Fourier transforms of polynomials and functions that grow like polynomials. One uses some ideas from functional analysis namely *duality arguments*. The class of functions we shall extend it to is the class $\mathcal{S}'(\mathbb{R})$ of all *tempered distributions* that we shall now define. In chapter 7 we talked about continuous maps between normed linear spaces. We shall now generalize this to include TVS such as $\mathcal{S}(\mathbb{R})$.

Definition (tempered distribution): A *tempered distribution* is a continuous linear map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$. The set of all such continuous linear maps denoted by $\mathcal{S}'(\mathbb{R})$ is a vector space known as the space of *tempered distributions.*

If V is a TVS then the set of all continuous linear maps $V \longrightarrow \mathbb{C}$ denoted by V^* is called the dual space of V. However it may happen that V^* consists of the zero element alone! An example is available on page 213 in the book *Goffman-Pedrick, A first course in functional analysis, Prentice Hall, 1965* cited earlier. Sixty years since its appearence this book is still a veritable Gem!

However this pathological behaviour is NOT exhibited by the space $\mathcal{S}(\mathbb{R})$. We shall produce several interesting elements in $\mathcal{S}'(\mathbb{R})$ and in fact $\mathcal{S}'(\mathbb{R})$ contains ALL the spaces $L^p(\mathbb{R})$. The secret for the richness of $\mathcal{S}'(\mathbb{R})$ lies in the fact that the space $\mathcal{S}(\mathbb{R})$ is a *locally convex TVS* and the *Hahn-Banach theorem* furnishes a rich supply of elements in the dual space.

The Dirac delta and its close cousins

1. Consider the map $\delta_0 : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ given by $f \mapsto f(0)$. This is evidently linear and its continuity with respect to the topology on $\mathcal{S}'(\mathbb{R})$ follows at once from the definition.

2. Suppose $Q(x)$ is a polynomial and $f \in \mathcal{S}(\mathbb{R})$ then $Q(x)f(x)$ is an element in $\mathcal{S}(\mathbb{R})$ and so is integrable. So we get the following

$$f \mapsto \int_{\mathbb{R}} Q(x)f(x)dx. \tag{10.3}$$

   Let us check that (10.3) is continuous. Well, owing to linearity we need to check that $f_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$ implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}} Q(x)f_n(x)dx \longrightarrow 0 \tag{10.4}$$

   Let the degree of $Q(x)$ be N and $R(x) = (1+x^2)^N Q(x)$. Then convergence of f_n gives us

$$\sup_{\mathbb{R}} |f_n(x)R(x)| \longrightarrow 0.$$

   Hence we get

$$|\int_{\mathbb{R}} Q(x)f_n(x)dx| \leq |\int_{\mathbb{R}} R(x)f_n(x)\frac{dx}{(1+x^2)^N}| \leq \sup_{\mathbb{R}} |f_n(x)R(x)| \int_{\mathbb{R}} \frac{dx}{(1+x^2)^N} \longrightarrow 0.$$

completing the proof.

3. Use the fact that $\mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$ for every $p \geq 1$ to show that if $F \in L^q(\mathbb{R})$ with $1 \leq q \leq \infty$, the map

$$f \mapsto \int_{\mathbb{R}} f(x)F(x)dx \tag{10.5}$$

defines a tempered distribution.

In these examples we say the functions $Q(x)$ and $F(x)$ represent the distributions (10.3) and (10.4) respectively. One then identifies the function with the associated distribution it defines namely (10.3) or (10.4). In future we shall make no distiction between an L^q function and the distribution it defines via (10.5). We still have to check one detail.

Classical functions regarded as distributions Suppose g and h both lie in $L^q(\mathbb{R})$ and for all $f \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_{\mathbb{R}} f(x)h(x)dx$$

then we have to show $g = h$ almost everywhere. Well, put $g(x) - h(x) = F(x)$ and what we have is that

$$\int_{\mathbb{R}} F(x)f(x)dx = 0, \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}). \tag{10.6}$$

To conclude that $F = 0$ almost everywhere, use standard techniques from measure theory. Consider the set $A = \{x \in \mathbb{R} : F(x) > 0\}$. Suppose A has positive measure. Take a compact subset $B \subset A$ of with positive measure. Assume $p \neq \infty$ and $p \neq 1$.

$$F^{p-1}\chi_B$$

is in $L^q(\mathbb{R})$. Select a sequence f_n in $\mathcal{S}(\mathbb{R})$ converging to $F^{p-1}\chi_B$ in $L^q(\mathbb{R})$. Then

$$|\int_{\mathbb{R}} F(x)(f_n(x) - F^{p-1}(x)\chi_B(x))dx| \leq \|F\|_p \|f_n - F^{p-1}(x)\chi_B(x)\|_q \longrightarrow 0.$$

Now using (10.6) this collapses to

$$\int_{\mathbb{R}} F^{p-1}(x)\chi_B(x)dx = 0.$$

This implies $F = 0$ on B. Contradiction. Since B has positive measure and $F > 0$ on B. There remain the cases $p = 1$ and $p = \infty$ which we leave to the audience as an exercise. The argument would have been much simpler in the case when F is continuous.

4. Consider the map $\delta_0' : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ given by $f \mapsto -f'(0)$. Show that this defines a tempered distribution (derivative of Dirac distribution).

5. Suppose u is a tempered distribution and $Q(x)$ is a polynomial, define $Q(x)u$ to be the map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ given by

$$f \mapsto u(Q(x)f), \quad f \in \mathcal{S}(\mathbb{R}) \tag{10.7}$$

Show that this defines a distribution and we say it is the product of u and the polynomial $Q(x)$. You need to prove the continuity of the map (10.7).

6. Show that $\cos(e^x)$ and $\sin(e^x)$ are tempered distribution. Is there a tempered distribution u such that

$$u(f) = \int_{\mathbb{R}} e^x f(x)dx, \quad f \in \mathcal{S}(\mathbb{R})? \text{ Ans: No.}$$

What about $e^x \cos(e^x)$ and $e^x \sin(e^x)$?

7. Show that for $f \in \mathcal{S}(\mathbb{R})$ the following limit exists

$$\lim_{\epsilon \to 0+} \int_{\mathbb{R}-[-\epsilon,\epsilon]} \frac{f(x)dx}{x}$$

and defines a tempered distribution known as $PV(\frac{1}{x})$