Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

54 Topology on the Schwartz space

X Brief introduction to tempered distributions

Ever since its inception in (1945) by *Laurant Schwartz*, distributions have played an increasingly important role in modern analysis. Here we shall discuss briefly some (very basic) aspects of distribution theory mainly in the context of Fourier analysis. A thorough account is clearly outside the scope of this course. Emphasis will be on basic examples and manipulations with distributions. We shall confine our attention to the so called *tempered distributions* which is relevant for Fourier analysis. For understanding the role of distributions in modern analysis the essay of *Lars Garding* in his book *Some points in analysis and its history, American mathematical society, Providence, Rhode Island (1997) pp. 77-87* must be consulted. As for the basic aspects of the subject we recommend two excellent introductory texts:

1. I. Richards and H. Youn, Distribution theory, a non-technical introduction, Cambridge University Press, 2007.

2. R. Strichartz, Guide to distributions theory and Fourier transforms, CRC Press, Boca Raton, 1994.

The space $\mathcal{S}(\mathbb{R})$ revisited We recall that $\mathcal{S}(\mathbb{R})$ is the vector space of rapidly decreasing functions that was introduced in chapter 4. Here we shall introduce a topology on this space. In fact it suffices to specify the convergent sequences and shall specify the metric only when needed.

Definition A sequence $\{f_n\}$ in $\mathcal{S}(\mathbb{R})$ converges to $f \in \mathcal{S}(\mathbb{R})$ if and only if for all $k, l \in \mathbb{N}$,

$$\sup_{\mathbb{R}} |x^k(D^l f_n(x) - D^l f(x))| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{10.1}$$

Clearly with this definition we see that the vector space operations on $\mathcal{S}(\mathbb{R})$:

$$(f, g) \mapsto f + g, \quad (\lambda, f) \mapsto \lambda f, \quad f, g \in \mathcal{S}(\mathbb{R}) \text{ and } \lambda \in \mathbb{R}.$$

are both (sequentially) continuous.

The TVS $\mathcal{S}(\mathbb{R})$: In other words the space $\mathcal{S}(\mathbb{R})$ is a topological vector space. It is a fact that this space is not normable namely the above notion of convergence DOES NOT arise out of a norm on $\mathcal{S}(\mathbb{R})$.
Exercises:

1. Check that the map $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ given by

$$(f, g) \mapsto fg, \quad f, g \in \mathcal{S}(\mathbb{R})$$

   is also continuous.

2. Examine whether $\exp(-x^2/n)$ converges in the space $\mathcal{S}(\mathbb{R})$. Ans: No.

3. What about the sequence $\exp(-nx^2)$?

4. Prove that convergence in $\mathcal{S}(\mathbb{R})$ implies convergence in L^2 norm.

5. Let $\phi \in C^\infty(\mathbb{R})$ such that $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq 2$ and $f \in \mathcal{S}(\mathbb{R})$. Show that $f_n(x) = f(x)\phi(x/n)$ is a sequence in $\mathcal{S}(\mathbb{R})$ converging to f in $\mathcal{S}(\mathbb{R})$. Thus the space of smooth functions with compact support is dense in $\mathcal{S}(\mathbb{R})$.

Recall that if $f \in \mathcal{S}(\mathbb{R})$ then $f' \in \mathcal{S}(\mathbb{R})$. Is the linear operator $f \mapsto f'$ continuous with respect to the topology on $\mathcal{S}(\mathbb{R})$? The following theorem addresses this issue.

Theorem (Continuity of differentiation): The differentiation map on $\mathcal{S}(\mathbb{R})$ given by $f \mapsto f'$ is continuous.

Well, let $f_n \to f$ in $\mathcal{S}(\mathbb{R})$ so that for all $k, l \in \mathbb{N}$ we have that

$$\sup_{\mathbb{R}} |x^k(D^l f_n(x) - D^l f(x))| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty. \tag{10.1}$$

Replacing l by $l+1$ we infer from (10.1) that for all $k, l \in \mathbb{N}$,

$$\sup_{\mathbb{R}} |x^k(D^l f_n'(x) - D^l f'(x))| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty. \tag{10.1$'$}$$

This completes the proof of the theorem.

Continuity of the Fourier transform We know that the Fourier transform maps $\mathcal{S}(\mathbb{R})$ onto itself. We have proved that this is continuous with respect to the L^2 norm. We now discuss the continuity of the Fourier transform with respect to the topology on $\mathcal{S}(\mathbb{R})$. Recall from chapter 4 that

$$|\xi^k D^l(\widehat{f}_n - \widehat{f})| = |\mathcal{F}(D^k(x^l(f_n - f)))(\xi)|$$

where $\mathcal{F}h$ is the Fourier transform of h. But now recall that

$$\sup_{\mathbb{R}} |\mathcal{F}(h)(\xi)| \leq \int_{\mathbb{R}} |h(x)| dx.$$

So that

$$\sup_{\mathbb{R}} |\xi^k D^l(\widehat{f}_n(\xi) - \widehat{f}(\xi))| \leq \int_{\mathbb{R}} |D^k(x^l(f_n(x) - f(x)))| dx$$

To estimate the integral we multiply and divide by $(1 + x^2)$ and we see that

$$\sup_{\mathbb{R}} |\xi^k D^l(\widehat{f}_n(\xi) - \widehat{f}(\xi))| \leq \sum_{s=0}^{k} \sum_{t=0}^{l+2} C_{s,t} \int_{\mathbb{R}} |x^t(D^s f_n(x) - D^s f(x))| \frac{dx}{1 + x^2}$$

where $C_{s,t}$ are constants independent of n. From this we conclude

$$\sup_{\mathbb{R}} |\xi^k D^l(\widehat{f}_n(\xi) - \widehat{f}(\xi))| \leq \pi \sum_{s=0}^{k} \sum_{t=0}^{l+2} C_{s,t} \sup_{\mathbb{R}} |x^t(D^s f_n(x) - D^s f(x))| \tag{10.2}$$

Now if $f_n \longrightarrow f$ in $\mathcal{S}(\mathbb{R})$ then the RHS of (10.2) goes to zero as $n \to \infty$ and we conclude that

$$\widehat{f}_n(\xi) \longrightarrow \widehat{f}(\xi)$$

in the space $\mathcal{S}(\mathbb{R})$ as $n \to \infty$. Let us record this as an important result:

Theorem: The Fourier transform $f \mapsto \widehat{f}$ as a linear operator on $\mathcal{S}(\mathbb{R})$ is continuous with respect to the topology of $\mathcal{S}(\mathbb{R})$.

Recall that in chapter 4 we extended the Fourier transform from $\mathcal{S}(\mathbb{R})$ to an operator on $L^2(\mathbb{R})$. Now is there a way to extend it any further ? For example can we extend it to all $L^p(\mathbb{R})$ and even beyond. We shall now see that this is possible. In fact one can even define the Fourier transform of functions like $\exp(ita)$ $(a \in \mathbb{R})$ which don't have any decay and even define Fourier transforms of polynomials and functions that grow like polynomials. One uses some ideas from functional analysis namely *duality arguments*. The class of functions we shall extend it to is the class $\mathcal{S}'(\mathbb{R})$ of all *tempered distributions* that we shall now define. In chapter 7 we talked about continuous maps between normed linear spaces. We shall now generalize this to include TVS such as $\mathcal{S}(\mathbb{R})$.