# Fourier Analysis and its Applications

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## 53 Dirichlet's theorem on Fourier series contd...

**Dirichlet's theorem on monotone functions:** Suppose $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a *piecewise continuous monotone function* the Fourier series of $f$ converges to $f(x)$ at all the points of continuity and at a point $x_0$ of discontinuity it converges to

$$\frac{1}{2}(f(x_0+) + f(x_0-)).$$

At the points $\pm\pi$ the Fourier series converges to

$$\frac{1}{2}(f(\pi-) + f(-\pi+)).$$

To simplify Dirichlet's argument *Ossian Bonnet* established the mean value theorem for integrals that bear his name.

**Bonnet's mean value theorem** Let $g : [a, b] \longrightarrow \mathbb{R}$ be continuous.

(i) If $f : [a, b] \longrightarrow \mathbb{R}$ is monotone then there exists $c \in [a, b]$ such that

$$\int_a^b f(t)g(t)dt = f(b-)\int_c^b g(t)dt + f(a+)\int_a^c g(t)dt.$$

(ii) If $f$ is monotone decreasing and $f \geq 0$ then for some $c \in [a, b]$

$$\int_a^b f(t)g(t)dt = f(a+)\int_a^c g(t)dt.$$

(iii) If $f$ is monotone increasing and $f \leq 0$ then for some $c \in [a, b]$

$$\int_a^b f(t)g(t)dt = f(b-)\int_c^b g(t)dt.$$

We insert here a preliminary result.

**Theorem**

(i) We have the result:

$$\int_0^\infty \frac{\sin \omega t}{t} dt = \operatorname{sgn}(\omega)\frac{\pi}{2}.$$

(ii) If $f : [0, \infty) \longrightarrow \mathbb{R}$ is monotone, for any $s > 0$,

$$\lim_{\omega\to\infty} \int_0^s f(u)\frac{\sin \omega u}{u} du = \frac{\pi}{2} f(0+).$$

**Theorem (Abel's summation by parts formula):** Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences then

$$\sum_{j=1}^{N} a_j(b_j - b_{j-1}) = -\sum_{j=1}^{N}(a_j - a_{j-1})b_{j-1} + a_N b_N.$$

with the understanding that $a_0 = b_0 = 0$.

Proof is of course quite clear. Observe the analogy with the rule for integration by parts.

**Proof of Bonnet's mean value theorem** Let us assume that $f$ is monotone decreasing. Since the product $fg$ is Riemann integrable, the idea is to approximate the integral of $fg$ by its Riemann sums, apply the rule for summation by parts and pass to the limit. The Riemann sums for the integral $\int f(t)g(t)dt$ are given by

$$\mathcal{R}(fg, \Pi) = \sum_{j=1}^{N} f(c_j)g(c_j)(t_j - t_{j-1})$$

where $\Pi$ denotes the partition of $[a, b]$, $\mathcal{R}$ denotes the Riemann sum and $c_j$ is specified below. Since $g$ is assumed to be continuous we have that

$$\int_{t_{j-1}}^{t_j} g(u)du = g(c_j)(t_j - t_{j-1})$$

for some $c_j \in (t_{j-1}, t_j)$. With this choice of $c_j$ for the Riemann sums displayed we get

$$\begin{aligned}
\mathcal{R}(fg, \Pi) &= \sum_{j=1}^{N} f(c_j) \int_{t_{j-1}}^{t_j} g(u)du \\
&= \sum_{j=1}^{N} f(c_j)\Big( \int_{a}^{t_j} g(u)du - \int_{a}^{t_{j-1}} g(u)du\Big) \\
&= -\sum_{j=1}^{N}(f(c_j) - f(c_{j-1})) \int_{a}^{t_{j-1}} g(u)du + f(c_n) \int_{a}^{t_n} g(u)du
\end{aligned}$$

As the mesh of the partition goes to zero, the last displayed term tends to $f(b-) \displaystyle\int_a^b g(u)du$. We now deal with the term $J = -\displaystyle\sum_{j=1}^{N}(f(c_j) - f(c_{j-1})) \int_{a}^{t_{j-1}} g(u)du$ Now the function

$$\phi(x) = -\int_{a}^{x} g(u)du$$

is continuous with respect to $x$ and let $M$ and $m$ be the supremum and infimum of $\phi$. Then

$$m \leq -\int_{a}^{t_{j-1}} g(u)du \leq M.$$

Multiplying by $f(c_j) - f(c_{j-1})$ and adding we get

$$m(f(c_n) - f(c_1)) \leq J \leq M(f(c_n) - f(c_1)).$$

Letting the mesh of the partition to tend to zero we get

$$m(f(b-) - f(a+)) \leq \lim J \leq M(f(b-) - f(a+))$$

Thus

$$\lim \Big(\frac{J}{f(b-) - f(a+)}\Big)$$

lies between $m$ and $M$ and so there is a $c \in (a, b)$ such that

$$\lim J = (f(b-) - f(a+)) \int_a^c g(u)du.$$

Hence we get in the limit

$$\int_a^b f(t)g(t)dt = -(f(b-) - f(a+)) \int_a^c g(u)du + f(b-) \int_a^b g(u)du.$$

whereby

$$\int_a^b f(t)g(t)dt = f(b-) \int_c^b g(u)du + f(a+) \int_a^c g(u)du$$

Now we turn to the other parts of Bonnet's theorem. From the equation

$$\int_a^b f(t)g(t)dt = f(b-) \int_c^b g(u)du + f(a+) \int_a^c g(u)du$$

subtract off

$$f(a+) \int_a^c g(u)du$$

and we get

$$\int_a^b (f(t) - f(a+))g(t)dt = (f(b-) - f(a+)) \int_c^b g(u)du$$

Now observe that $f(x) - f(a+)$ is decreasing and non-negative and we get the second part of Bonnet's theorem. Write $F(t) = f(t) - f(a+)$. The third part is similar. Proof of Bonnet's theorem is now complete.

**Proof of Dirichlet's theorem** We now turn to the proof of Dirichlet's theorem. Recall that

$$S_n(f, x) = \int_{-\pi}^{\pi} D_n(t)f(x - t)dt$$

which in view of the fact that the integrand is $2\pi$-periodic, can be written as

$$S_n(f, x) = \int_{x-\pi}^{x+\pi} D_n(t)f(x - t)dt$$

Again writing out the Dirichlet kernel and using Riemann Lebesgue lemma, finding the limit of $S_n(f, x)$ is the same as finding the limit of

$$\frac{1}{2\pi} \int_{x-\pi}^{x+\pi} \sin nt \cot(t/2)f(x - t)dt$$

Next, writing $\cot(t/2) = (\cot(t/2)-2/t)+(2/t)$. Since $\cot(t/2)-2/t$ is continuous, Riemann Lebesgue lemma allows us to conclude that the limit of $S_n(f, x)$ as $n \to \infty$ is the same as the limit of the expression

$$\frac{1}{\pi} \int_{x-\pi}^{x+\pi} \Big(\frac{\sin nt}{t}\Big) f(x - t)dt$$

Write

$$[-\pi, \pi] = I_1 \cup I_2 \cup \cdots \cup I_k$$

where $I_1, I_2, \ldots, I_k$ are non-overlapping intervals and $f$ restricted to each $I_j$ is continuous and monotone.

Assume that $x \notin I_j = [c, d]$ so that $x - c$ and $x - d$ are both non zero and have the same sign and so

$$\frac{1}{\pi} \int_{x-I_j} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt = \pm \frac{1}{\pi} \int_{x-d}^{x-c} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt$$

Put $nt = u$ and we get

$$\frac{1}{\pi} \int_{x-I_j} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt = \pm \frac{1}{\pi} \int_{n(x-d)}^{n(x-c)} \Big( \frac{\sin u}{u} \Big) f(x - \frac{u}{n}) dt$$

Letting $n \longrightarrow \infty$ we see that

$$\frac{1}{\pi} \int_{x-I_j} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt \longrightarrow 0, \quad n \to \infty$$

in this case. Next, if $x \in I_l = [a, b]$ then $x - b \leq 0 \leq x - a$ and not both zero. We split the integral on $x - I_l$ into a sum of two integrals over $[x - b, 0]$ and $[0, x - a]$.

$$\pi \int_{x-I_l} = \int_{x-b}^{0} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt + \int_{0}^{x-a} \Big( \frac{\sin nt}{t} \Big) f(x-t) dt$$

This time the substitution $nt = u$ gives

$$\pi \int_{x-I_l} = \int_{n(x-b)}^{0} \Big( \frac{\sin u}{u} \Big) f(x - \frac{u}{n}) dt + \int_{0}^{n(x-a)} \Big( \frac{\sin u}{u} \Big) f(x - \frac{u}{n}) dt$$

If $x - a$ or $x - b$ is zero one of the integrals is zero. To deal with the limit when $n \to \infty$ we have to use theorem 103 namely when $f$ is monotone,

$$\lim_{\omega \to \infty} \int_0^s f(u) \frac{\sin \omega u}{u} du = \frac{\pi}{2} f(0+).$$

So we get in the limit the precise value

$$\frac{\pi}{2} \Big( f(x-) + f(x+) \Big)$$

One of these is to be omitted if $x - a$ or $x - b$ is zero that is if $x$ is an endpoint of $I_l$. In such a case there will obviously be a non-zero contribution from two adjacent intervals. Now since

$$S_n(f, x) = \int_{x-\pi}^{x+\pi} = \sum_{j=1}^{n} \int_{x-I_j}$$

and all but one/two integrals contribute as $n \to \infty$ we get the result that

$$\lim_{n \to \infty} S_n(f, x) = \frac{\pi}{2} \Big( f(x-) + f(x+) \Big)$$

The proof of Dirichlet's theorem is now complete including all the auxillary propositions !