Fourier Analysis and its Applications

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52 Dirichlet's theorem on Fourier series

Dirichlet's theorem on monotone functions. Let us recall some properties of monotone functions.

(i) A monotone function has only countably many discontinuities.

(ii) The discontinuities are simple jump discontinuities.

(iii) A monotone function is Riemann integrable.

(iv) Given any countable dense subset D there is a monotone function which is discontinuous precisely at points of D.

(v) A monotone function is differentiable almost everywhere.

Of these the last property is highly non-trivial (See for example *Royden's Real Analysis*). The idea behind stating this list is that monotone functions can be quite badly behaved in view of (iv). For example their discontinuities could be dense. Nevertheless we shall see that the Fourier series of a piecewise continuous monotone function is quite well behaved as regards convergence. This is the theorem of Dirichlet that we shall establish presently.

Properties of monotone functions To prove (ii), let $f : [a, b] \longrightarrow \mathbb{R}$ be monotone increasing and $c \in [a, b]$. We set

$$l = \inf\{f(t) \ : \ t \in (c, b]\}.$$

and we show that $f(t) \longrightarrow l$ as $t \to c+$ which proves the existence of the right limit $f(c+)$. So let $\epsilon > 0$ be arbitrary. There exists $t_0 \in (c, b]$ such that

$$f(t_0) < l + \epsilon.$$

Then by monotonocity,

$$l \leq f(t) < l + \epsilon, \quad c < t < t_0.$$

So the requisite $\delta > 0$ is $t_0 - c$. The proof of the existence of a limit from the left is similar. We also see that

$$f(c-) \leq f(c+).$$

Now suppose $a \leq c_1 < c_2 \leq b$. Then trvially,

$$\inf\{f(t) \ : \ t \in (c_1, c_2)\} \leq \sup\{f(t) \ : \ t \in (c_1, c_2\}$$

whereby we conclude

$$f(c_1-) \leq f(c_1+) \leq f(c_2-) \leq f(c_2+)$$

Thus if c_1 and c_2 are two points of discontinuity then the intervals

$$I_{c_1} = (f(c_1-), f(c_1+)), \quad I_{c_2} = (f(c_2-), f(c_2+))$$

are pairwise disjoint non-empty open intervals. So if for each discontinuity c we pick a rational number

$$Q(c) \in (f(c-), f(c+))$$

then the function $c \mapsto Q(c)$ from the set E of discontinuities of f into $\mathbb{Q}$ is injective and so the *discontinuities can be at most countable.* From this the Riemann integrability of a monotone function follows at once. Next suppose E is any countable dense subset of $[a, b]$ with enumeration

$$E = \{r_1, r_2, r_3, \dots, \}$$

We select any convergent series of positive terms say $\sum 2^{-n}$. We construct our function $f : [a, b] \longrightarrow \mathbb{R}$ as follows:

$$f(x) = \sum_{r_j < x} 2^{-j}$$

This function is evidently monotone. If $x < y$ then there are more indices j such that $r_j < y$ than $r_j < x$. It is not difficult to verify that the function is continuous on $[a, b] - E$ and discontinuous on E. We leave the details to the audience.

Dirichlet's theorem (1829) is the earliest rigorous result of a general kind for piecewise continuous monotone functions.

Dirichlet's theorem on monotone functions: Suppose $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a *piecewise continuous monotone function* the Fourier series of f converges to $f(x)$ at all the points of continuity and at a point x_0 of discontinuity it converges to

$$\frac{1}{2}(f(x_0+) + f(x_0-)).$$

At the points $\pm\pi$ the Fourier series converges to

$$\frac{1}{2}(f(\pi-) + f(-\pi+)).$$

To simplify Dirichlet's argument *Ossian Bonnet* established the mean value theorem for integrals that bear his name.

Bonnet's mean value theorem Let $g : [a, b] \longrightarrow \mathbb{R}$ be continuous.

(i) If $f : [a, b] \longrightarrow \mathbb{R}$ is monotone then there exists $c \in [a, b]$ such that

$$\int_a^b f(t)g(t)dt = f(b-)\int_c^b g(t)dt + f(a+)\int_a^c g(t)dt.$$

(ii) If f is monotone decreasing and $f \geq 0$ then for some $c \in [a, b]$

$$\int_a^b f(t)g(t)dt = f(a+)\int_a^c g(t)dt.$$

(iii) If f is monotone increasing and $f \leq 0$ then for some $c \in [a, b]$

$$\int_a^b f(t)g(t)dt = f(b-)\int_c^b g(t)dt.$$

We insert here a preliminary result.

Theorem

(i) We have the result:

$$\int_0^\infty \frac{\sin \omega t}{t} dt = \operatorname{sgn}(\omega)\frac{\pi}{2}.$$

(ii) If $f : [0, \infty) \longrightarrow \mathbb{R}$ is monotone, for any $s > 0$,

$$\lim_{\omega\to\infty} \int_0^s f(u)\frac{\sin \omega u}{u} du = \frac{\pi}{2} f(0+).$$

Proof: The first part is standard. It is obviously enough to prove it for $\omega > 0$. Make the change of variables $t\omega = u$ and we are led to proving

$$\int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}.$$

This is equivalent to

$$\int_{-\infty}^\infty \frac{\sin u}{u} du = \pi$$

which is easily done using Cauchy's theorem applied to $(e^{iz} - 1)/z$ on a semi-circular contour indented at the origin. We leave the details for the audience since this is probably familar from courses on complex analysis.

Turning to the proof of (ii) we write (*we may clearly assume f is monotone decreasing and non-positive*)

$$\int_0^s f(u)\frac{\sin \omega u}{u} du = \int_0^s (f(u) - f(0+))\frac{\sin \omega u}{u} du + f(0+) \int_0^s \frac{\sin \omega u}{u} du$$

Put $\omega u = t$ in the second term and let $\omega \to \infty$ and we get $\pi f(0+)/2$. We now tackle the first term. Let $\epsilon > 0$ be arbitrary. Select $r > 0$ such that

$$|f(u) - f(0+)| < \epsilon/A, \quad 0 < u \le r.$$

where A is a constant that will be specified. Then

$$\int_r^s \Big(\frac{f(u) - f(0+)}{u}\Big) \sin \omega u du \longrightarrow 0$$

by Riemann Lebesgue lemma and we are left with

$$\int_0^r \Big(\frac{f(u) - f(0+)}{u}\Big) \sin \omega u du$$

to which we apply the mean value theorem of Bonnet leading to

$$\int_0^r \Big(\frac{f(u) - f(0+)}{u}\Big) \sin \omega u du = (f(r) - f(0+)) \int_c^r \frac{\sin \omega u}{u} du$$

for some $c \in [0, r]$. Since the integral term is bounded in absolute value by A say. We see that

$$\int_0^r \Big(\frac{f(u) - f(0+)}{u}\Big) \sin \omega u du$$

is less than ϵ in absolute value. The proof of the theorem is thereby completed.

We shall now prove the theorem of Bonnet since this theorem is usually not available in modern standard texts. A proof is available in *G. A. Gibson's Advanced Calculus, p. 277.* For convenience we state it here again:

If $f : [a, b] \longrightarrow \mathbb{R}$ monotone and $g : [a, b] \longrightarrow \mathbb{R}$ continuous then there is a $c \in [a, b]$ such that

$$\int_a^b f(t)g(t)dt = f(b-)\int_c^b g(t)dt + f(a+)\int_a^c g(t)dt.$$

Proof of Bonnet's mean value theorem is a beautiful application of Abel's summation by parts formula. Since this formula is used frequently towards the analysis of conditionally convergent series (and hence the Fourier series of many interesting functions) we state it here.

Theorem (Abel's summation by parts formula): Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences then

$$\sum_{j=1}^{N} a_j(b_j - b_{j-1}) = -\sum_{j=1}^{N}(a_j - a_{j-1})b_{j-1} + a_N b_N.$$

with the understanding that $a_0 = b_0 = 0$.

Proof is of course quite clear. Observe the analogy with the rule for integration by parts.