Fourier Analysis and its Applications

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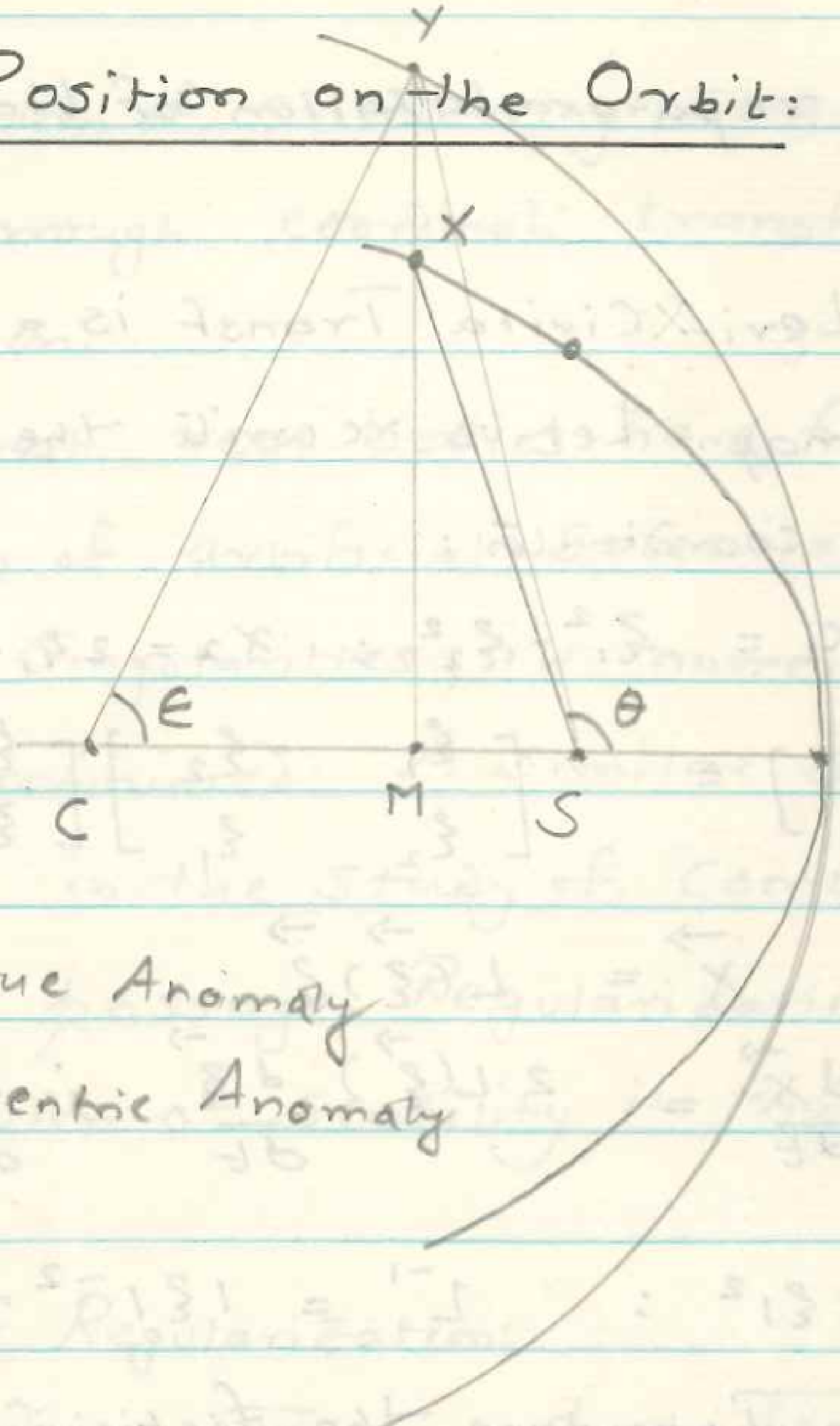
50 Inverting the Kepler Equation using Fourier Analysis

The Kepler problem Recall that the perihelion is that point on the orbit of the planet which is closest to the sun. Suppose the planet passes the perihelion P at time $t = 0$. Placing the sun at the focus S, we measure all angles from the radius vector SP. Let $X(t)$ be the position of the planet at time t and $\theta(t)$ be the angle between SP and SX. In astronomy one calls the function $\theta(t)$ the *True Anomaly.*

Kepler Problem: Find the function $\theta(t)$ as explicitly as possible.

PICTURE

§9.6 Position on the Orbit: (The Kepler Eqⁿ)



θ = true Anomaly

E = Eccentric Anomaly

A basic lemma: Let X and Y be corresponding points on the ellipse and the auxiliary circle. Then Area(PSX) refers to the area of the sector of the ellipse and Area(PSY) the corresponding sector of the auxiliary circle.

$$\frac{Area(PSX)}{Area(PSY)} = \frac{b}{a} = \frac{Area(Ellipse)}{Area(Circle)} \tag{8.11}$$

Exercise:

1. Prove this using integral calculus to compute the indicated areas.

We now use *Kepler's second law* and write

$$\frac{Area(PSX)}{Area(Ellipse)} = \frac{t}{T} = \frac{Area(PSY)}{Area(Circle)} \tag{8.12}$$

But if C denotes the center of the ellipse (and the circle) then

$$Area(PSY) = Area(PCY) - Area(\Delta SCY) \tag{8.13}$$

The last two areas are readily described and we get

$$Area(PSY) = \frac{1}{2}(a^2E - a^2\epsilon \sin E) \tag{8.14}$$

where E is the eccentric angle of P (namely, $\angle PCY$) and ϵ the eccentricity of the ellipse.

So from (8.14) we get

$$\frac{Area(PSY)}{Area(Circle)} = \frac{E - \epsilon \sin E}{2\pi}$$

or, using (8.12) we get

$$E - \epsilon \sin E = \frac{2\pi t}{T} \tag{8.15}$$

Equation (8.15) is the famous *Kepler Equation*. The number $2\pi t/T$ is called the *Mean Anomaly*.

Inverting the Kepler equation Exercises:

2. Show that the function $E - \epsilon \sin E$ is strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$. The inverse function $E(t)$ is a strictly increasing infinitely differentiable function.

3. Explain why the function $E(t)$ is an odd function. Use both geometrical reasoning as well as mathematical analysis.

4. Show that $E(0) = 0$ and $E(T/2) = \pi$.

The problem of inverting the Kepler equation has been studied by many eminent mathematicians such as *J. L. Lagrange, Memoirs of the Berlin Academie 1768-69. Also volume - II, p. 22 ff. of his Méchanique Analytique 1815¿.*

In this connection Lagrange discovered the *inversion formula* that bears his name. The *Lagrange inversion formula* has important applications in quite un-related fields such as combinatorics.

Bringing in the periodicity Let us now look at $E(t+T)$. The Kepler equation gives

$$E(t+T) - \epsilon \sin E(t+T) = \frac{2\pi(t+T)}{T} = \frac{2\pi t}{T} + 2\pi.$$

which can be written as

$$\begin{aligned} E(t+T) - \epsilon \sin E(t+T) &= (E(t) - \epsilon \sin E(t)) + 2\pi \\ &= (E(t) + 2\pi) - \epsilon \sin(E(t) + 2\pi). \end{aligned}$$

By injectivity of the function $\lambda \mapsto \lambda - \epsilon \sin \lambda$ we conclude

$$E(t+T) = E(t) + 2\pi. \tag{8.16}$$

Exercises:

5. Could you derive this directly from physical considerations?
6. Show using (8.16) that $\psi(t) = E(t) - \frac{2\pi t}{T}$ is a periodic function with period T.

Clearly then $\psi\left(\frac{tT}{2\pi}\right)$ is periodic with period 2π. Let us write the Fourier series for $\psi\left(\frac{tT}{2\pi}\right)$:

$$\psi\left(\frac{tT}{2\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nt),$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \psi\left(\frac{tT}{2\pi}\right)\sin(nt)dt = \frac{2}{\pi}\int_0^{\pi}\left\{E\left(\frac{tT}{2\pi}\right) - t\right\}\sin(nt)dt.$$

Integrating by parts and recalling $E(T/2) = \pi$, $E(0) = 0$, we get

$$b_n = \frac{2}{n\pi}\int_0^{\pi}\frac{d}{dt}\left\{E\left(\frac{tT}{2\pi}\right) - t\right\}\cos(nt)dt.$$

This simplifies to

$$b_n = \frac{2}{n\pi}\int_0^{\pi}\frac{d}{dt}E\left(\frac{tT}{2\pi}\right)\cos(nt)dt = \frac{2}{n\pi}\int_0^{T/2} E'(s)\cos(2\pi ns/T)ds.$$

Using the Kepler equation again, the argument of the cosine can be re-written resulting in:

$$b_n = \frac{2}{n\pi}\int_0^{T/2} E'(s)\cos(nE(s) - n\epsilon\sin E(s))ds.$$

The change of variables $E(s) = \lambda$ now gives

$$b_n = \frac{2}{n\pi}\int_0^{\pi}\cos(n\lambda - n\epsilon\sin\lambda)d\lambda = \frac{2J_n(n\epsilon)}{n}.$$

The Fourier series now reads

$$\psi\left(\frac{tT}{2\pi}\right) = E\left(\frac{tT}{2\pi}\right) - t = \sum_{n=1}^{\infty}\frac{2J_n(n\epsilon)}{n}\sin nt.$$

So the eccentric angle $E(t)$ (also known in astronomy as the *Eccentric Anomaly*) can be written as a *Kaypten series*:

$$E(t) = \frac{2\pi t}{T} + \sum_{n=1}^{\infty}\frac{2J_n(n\epsilon)}{n}\sin(2\pi nt/T). \tag{8.17}$$

The true anomaly

7. Using elementary trigonometry, find a relation between the true anomaly and the eccentric anomaly.

The relation is

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\left(\frac{E}{2}\right).$$

For a short but quick historical survey see C. A. Ronan, Science, its history and development among world's culture, pp. 336-337. The book of D. C. Knight, Johannes Kepler and planetary motion, Chatto and Windus, London 1965, contains a poignant account of the life and times of the great astronomer and mathematician.
For more on the Kepler problem and mathematical principles underlying celestial mechanics, the book by *J. M. Danby, Celestial Mechanics and dynamical astronomy, Kluwer Academic, 1991*, is HIGHLY recommended. This second edition contains computer experiments.
A more ambitious project would be to read the comprehensive two volumes
D. Boccaletti and G. Puccaco, Theory of Orbits, Vol- I, II, Springer Verlag, 2004.

One can try to expand the Bessel functions appearing in the Kaypten series

$$E(t) = \frac{2\pi t}{T} + \sum_{n=1}^{\infty} \frac{2J_n(n\epsilon)}{n} \sin(2\pi n t/T). \tag{8.17}$$

and rearrange terms to get a power series in ϵ. however it was known that the resulting series converges only when $\epsilon < 0.667$. The orbits of most comets exceed this number - Orbit of *Halley's comet* is 0.96 !!

It seems an investigation into why the series fails to converge beyond this threshold led Cauchy to develop the theory of functions of one complex variable. There is an imaginary singularity of the solution of Kepler equation

$$E(t) - \epsilon \sin E(t) = \frac{2\pi t}{T}$$

that prevents the power series from converging beyond the threshold value.