Fourier Analysis and its Applications

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05 The Jacobi theta function identity

Jacobi Theta Function Identity We shall now give another application of the basic convergence theorem and derive a beautiful identity that is of immense use in *number theory* . Let us consider the function:

$$f(t) = \sum_{n=-\infty}^{\infty} \exp(-(t+2\pi n)^2) \tag{1.23}$$

Exercises: (i) Show that this function is infinitely differentiable.

(ii) We need to interchange limits and integrals. Look up the conditions permitting this in *Rudin's Principles of Math. Analysis* .

So this is a smooth even 2π periodic function of t and we can apply the basic convergence theorem. Let us compute the Fourier coefficients of this function.

$$2\pi a_0 = \int_{-\pi}^{\pi} f(t)dt = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \exp(-(t+2\pi n)^2)dt$$

Put $t + 2\pi n = u$ in the integral and we get

$$2\pi a_0 = \sum_{n=-\infty}^{\infty} \int_{\pi(2n-1)}^{\pi(2n+1)} \exp(-u^2)du = \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Exercise: Compute the Fourier coefficients a_n using (1.22). Obviously $b_n = 0$ for all n. Check that the Fourier series for $f(t)$ is given by

$$f(t) = \frac{1}{2\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp(-n^2/4) \cos nt.$$

So we have obtained the identity:

$$\sum_{n=-\infty}^{\infty} \exp(-(t+2\pi n)^2) = \frac{1}{2\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp(-n^2/4) \cos nt \tag{1.24}$$

Here is a more general version that we leave as an exercise (following exactly the same steps):

Theorem:

$$\sum_{n=-\infty}^{\infty} \exp(-\tau(t+2\pi n)^2) = \frac{1}{2\sqrt{\tau\pi}} \Big(1 + 2\sum_{n=1}^{\infty} \exp(-n^2/4\tau) \cos nt\Big) \tag{1.25}$$

Putting $t = 0$ and $4\tau = u$ we get

$$\sum_{n=-\infty}^{\infty} \exp(-\pi^2 n^2 u) = \frac{1}{\sqrt{u\pi}} \sum_{n=-\infty}^{\infty} \exp(-n^2/u) \tag{1.26}$$

The idea is to compute the Fourier series and apply the basic convergence theorem to the 2π periodic smooth function

$$\sum_{n=-\infty}^{\infty} \exp(-\tau(t+2\pi n)^2)$$

Definition: The function

$$\vartheta_0(\tau) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 \tau), \quad \text{Re}\,\tau > 0. \tag{1.27}$$

is called the *theta function of Jacobi* and the identity we have derived is equivalent to

$$\vartheta_0(\tau) = \frac{1}{\sqrt{\tau}} \vartheta_0\Big(\frac{1}{\tau}\Big) \tag{1.28}$$

The function $\vartheta_0(\tau)$ is holomorphic in the right half plane and $\sqrt{\tau}$ appearing in equation (1.28) is defined as

$$\sqrt{\tau} = \sqrt{|\tau|}\exp(\frac{i}{2}\text{Arg}\,\tau).$$

Arg τ is the principal value namely, in this case taking its value in $(-\pi/2, \pi/2)$.

For future use let us record the behaviour of $\vartheta_0(\tau)$ near infinity and zero (for *real values* of τ).

(1) The behaviour near infinity is easy:

$$\vartheta_0(\tau) = 1 + 2\exp(-\pi\tau)O(1), \qquad \text{as } \tau \longrightarrow \infty. \tag{1.29}$$

(2) For the behaviour near zero we need the functional identity (1.28). For τ close to the origin, $1/\tau >> 1$ and so using (1.29)

$$\vartheta_0(\tau) = \frac{1}{\sqrt{\tau}} + o(1), \qquad \text{as } \tau \longrightarrow 0+. \tag{1.30}$$

We shall need these to handle exchange of integrals and infinite sums and also understand the scope of the constraints imposed on the variables.

Riemann's functional function The theta function identity (1.28) is equivalent to the famous *functional equation of Riemann* which we state next. Recall that the zeta function $\zeta(z)$ is defined on Re $z > 1$ as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{1.31}$$

Theorem The function $\zeta(z)$ extends analytically to $\mathbb{C} - \{1\}$ and satisfies the *functional equation* :

$$\pi^{-z/2}\Gamma\Big(\frac{z}{2}\Big)\zeta(z) = \pi^{(1-z)/2}\Gamma\Big(\frac{1-z}{2}\Big)\zeta(1-z), \quad z \neq 0, 1 \tag{1.32}$$

We shall derive this celebrated result of B. Riemann following the treatment in *Richard Bellman, A brief introduction to theta functions, Holt Reinhart and Winston, New York, 1961.* (p. 31 ff).

Proof: We begin with a simple observation:

$$\int_0^\infty e^{-n^2\pi t} t^{s-1} dt = \frac{\Gamma(s)}{n^{2s}\pi^s}, \quad \text{Re } s > 0. \tag{1.33}$$

Setting $n = 1, 2, 3, \dots$ and summing,

$$\int_0^\infty \Big(\sum_{n=1}^\infty e^{-n^2\pi t}\Big) t^{s-1} dt = \Gamma(s)\zeta(2s)\pi^{-s}, \quad \text{Re } s > \frac{1}{2}. \tag{1.34}$$

As an exercie, justify the exchange of the summation and integration in the above derivation. Note that by virtue of our estimates (1.29)-(1.30) the integral on the left hand side of (1.34) converges if Re $s > \frac{1}{2}$ while the series on the right hand side converges for the same range of s. The summation appearing within the integral is of course related to the theta function and we must invoke the functional equation (1.28) for the theta function that we have derived. Replacing s by $s/2$ and using the notation

$$g(t) = \sum_{n=1}^\infty e^{-n^2\pi t} = \frac{1}{2}(\vartheta_0(t) - 1) \tag{1.35}$$

we get

$$\int_0^\infty g(t) t^{\frac{s}{2}-1} dt = \Gamma\Big(\frac{s}{2}\Big)\zeta(s)\pi^{-s/2}, \quad \text{Re } s > 1. \tag{1.36}$$

We break the integral on the left and rewrite (1.36) as

$$\int_0^1 g(t) t^{\frac{s}{2}-1} dt + \int_1^\infty g(t) t^{\frac{s}{2}-1} dt = \Gamma\Big(\frac{s}{2}\Big)\zeta(s)\pi^{-s/2}, \quad \text{Re } s > 1. \tag{1.37}$$

Now we use the functional equation (1.28) for the theta function to re-write (1.35) as

$$g(t) = -\frac{1}{2} + \frac{1}{2\sqrt{t}} + \frac{1}{\sqrt{t}} g\Big(\frac{1}{t}\Big). \tag{1.38}$$

and substituting this in the intergal over $[0, 1]$ appearing in (1.37), we get

$$\frac{1}{s(s-1)} + \int_0^1 g\Big(\frac{1}{t}\Big) t^{\frac{s}{2}-\frac{3}{2}} dt + \int_1^\infty g(t) t^{\frac{s}{2}-1} dt = \Gamma\Big(\frac{s}{2}\Big)\zeta(s)\pi^{-s/2}, \quad \text{Re } s > 1. \tag{1.39}$$

Exercises: (1) Show that the integral

$$\int_0^1 g\Big(\frac{1}{t}\Big) t^{\frac{s}{2}-\frac{3}{2}} dt = \int_1^\infty g(u) u^{-\frac{1}{2}-\frac{s}{2}} du \tag{1.40}$$

is an entire function of s. Show that the following integral appearing in (1.39)

$$\int_1^\infty g(t) t^{\frac{s}{2}-1} dt$$

is holomorphic in the half plane Re $s > 0$ thereby the left hand side of (1.39) provides an analytic continuation on $\zeta(s)$ on the punctured half plane Re $s > 0$ and $s \neq 1$. Finally let us use (1.40) in (1.39) and rewrite (1.39) in the elegant form:

$$-\frac{1}{s(1-s)} + \int_1^\infty g(t)\Big(t^{-\frac{1-s}{2}} + t^{-\frac{s}{2}}\Big)\frac{dt}{\sqrt{t}} = \Gamma\Big(\frac{s}{2}\Big)\zeta(s)\pi^{-s/2}, \quad \text{Re } s > 1. \tag{1.41}$$

The left hand side of (1.41) is invariant with respect to the substitution $s \mapsto 1 - s$ and we get as a consequence that on the strip $0 < \mathrm{Re}\, s < 1$, we have the celebrated *functional equation of B. Riemann* :

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s). \tag{1.42}$$

Of course to establish (1.42) for all $s \in \mathbb{C} - \{0, 1\}$ we need to analytically continue $\zeta(s)$ as a holomorphic function on $\mathbb{C} - \{1\}$. This is quite easy and is indicated as a set of exercises.