Fourier Analysis and its Applications

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VIII - Application to Celestial Mechanics

I know that I am mortal by nature, and ephemeral; but when I trace at my pleasure the windings to and fro of the heavenly bodies I no longer touch the earth with my feet: I stand in the presence of Zeus himself and take my fill of ambrosia. Famous epigram in Ptolemy's Almagest (AD 145). See the

interesting article *O. Gingerich, Was Ptolemy a fraud? Quarterly journal of the Royal Astronomical Society* **21** *(1980) 253-266.*

In this last chapter of this course we discuss the original problem that led Bessel in 1824 to introduce the functions that bear his name. Wilhelm Bessel was an astronomer at Königsberg and his chief interest was the study of orbits of comets.

We must begin by recalling the three basic laws of planetary motion enunciated by Johannas Kepler. The discovery of these laws forms an interesting culmination of classical astronomy. Equipped with his calculus, Issac Newton, with his laws of dynamics able to explain:

1. The motion of planets.

2. The precession of equinoxes.

3. The formation of tides.

Astronomy hitherto an *empirical science* transformed into a *dynamical science.*

Kepler's laws of planetary motion The first two laws were enunciated in 1609 in *De Motibus Stellae Martis*:

Kepler's First Law: The planets move around the sun in elliptical orbits with the sun at one of the foci.

Kepler's Second Law: The radius vector joining the sun and the planet sweeps out equal areas in equal intervals of time. This is just a restatement of the law of conservation of angular momentum.

The third law (which is an approximate law neglecting the masses of planets) appeared much later in 1619 in his *Harmonices Mundi.*

Kepler's Third Law: The square of the period T is proportional to the cube of the semi-major axis a of the orbit.

The constant of proportionality was determined by Gauss in his famous book on Astronomy *Theoria Motus Corporum Coelestium* and is known as Gaussian gravitational constant.

The problem of inversion in completely integrable systems: We first set the Kepler problem against the general back-drop of the theory of (non-linear) ODEs. Consider the system of ODEs in $\mathbb{R}^n$:

$$\frac{dy_j}{dt} = f_j(y_1, y_2, \ldots, y_n), \quad j = 1, 2, \ldots, n. \tag{8.1}$$

where the functions f_j are smooth functions and for simplicity we assume that they are independent of time t.

Solving the system (8.1) for a given system for a given set of initial conditions $y_j(0) = c_j$ means finding a curve in $\mathbb{R}^n$.

$$t \mapsto (y_1(t), y_2(t), \ldots, y_n(t))$$

to illustrate the procedure for finding the curve (8.1) let us look at the simplest case of a harmonic oscillator in $\mathbb{R}^2$:

$$\dot{x} = y, \quad \dot{y} = -x. \tag{8.2}$$

It is easy to obtain from (8.2) that

$$x^2 + y^2 = \text{const}$$

where the constant can be fixed using initial conditions. The function $x^2 + y^2$ obtained above is called a *first intgeral*. Let us define it precisely in general terms.

Definition: A non-constant smooth function $\phi(y_1, y_2, \ldots, y_n)$ is said to be a *first integral* for the system (8.1) if

$$\phi(y_1(t), y_2(t), \ldots, y_n(t)) \tag{8.3}$$

is constant along every solution $(y_1(t), y_2(t), \ldots, y_n(t))$ of the ODE. Equivalently, differentiating (8.3) if and only if $\nabla\phi \cdot f = 0$ where $f = (f_1, f_2, \ldots, f_n)$.

Use of first integrals Suppose we have a first integral ϕ then we know that the solution we are seeking lies on the $(n-1)$-dimensional hypersurface

$$\phi(y_1, y_2, \ldots, y_n) = \text{const} \tag{8.4}$$

where the constant is fixed using the intial conditions. *In other words our search for the solutions is now no longer in the n-dimensional space $\mathbb{R}^n$ but in an $(n-1)$-dimensional entity namely the locus (8.4)* or we have reduced the problem to an $(n-1)$-dimensional problem.

Let us illustrate this using two examples. For the case of the harmonic oscillator we have $x^2+y^2 = c$ which means the trajectory lives on a circle but this circle has infinitely many parametrizations but one and only one of these would qualify to be a solution of the ODE.

Which one? To answer this question let us put $x = \cos F(t)$ and $y = \sin F(t)$ where F is to be determined. Substituting into the ODES $\dot{x} = y, \dot{y} = -x$ we get

$$\dot{F}(t) = -1.$$

or $F(t) = -t + \alpha$ and we get the solution $x = \cos(t-\alpha), y = -\sin(t-\alpha)$ as expected.

A more sophisticated example Consider Euler's equations for a spinning top:

$$A\dot{x} = (B-C)yz, \quad B\dot{y} = (C-A)xz, \quad C\dot{z} = (A-B)xy. \tag{8.5}$$

We immediately see that this has two first integrals

$$Ax^2 + By^2 + Cz^2, \quad A^2x^2 + B^2y^2 + C^2z^2 \tag{8.6}$$

and so the solution lies on the intersection of the two ellipsoids

$$Ax^2 + By^2 + Cz^2 = \alpha, \quad A^2x^2 + B^2y^2 + C^2z^2 = \beta \tag{8.7}$$

The constants α and β are determined via initial conditions. If the gradients of these functions are linearly independent along the intersection *then the intersection is a smooth curve and this is the solution curve we are looking for !* except that this curve has infinitely many parametrizations and **which one of these qualifies to be the solution of the ODE?**

To understand this let us write

$$x^2 + y^2 + z^2 = u \tag{8.8}$$

and regard u as a variable. Solving (8.7)-(8.8) we get

$$x = F(u), \quad y = G(u), \quad z = H(u) \tag{8.9}$$

where F, G, H are known functions of u and now we substitute (8.9) into the system of ODEs and we get

$$F'(u)\frac{du}{dt} = \frac{B-C}{A}G(u)H(u).$$

This is a first order scalar ODE of the form

$$\dot{u} = \psi(u)$$

which can be integrated as

$$t = \int_{u(0)}^{u} \frac{ds}{\psi(s)} \tag{8.10}$$

After performing the integration, to get u as a function of t we need to invert the equation (8.10) and this is still a non-trivial problem. Already we see that the innocent looking problem in three dimensions requires some substantial work.

In the case of planetary motion it turns out that in this LAST phase of the problem Fourier analysis enters ! The equation to be inverted is called the Kepler equation.

Generally if we have a system of n-differential equations (8.1) and we have a system of $(n-1)$-first integrals with linearly independent gradients along their common locus, the intersection of the level sets is a curve and this is the solution curve except for parametrization. *Finding the correct parametrization involves one integration and one inversion.*

How do we get first integrals? There are two common sources for finding first intgerals:

(i) Physics provides us with a supply of first integrals such as the law of conservation of energy for instance.

(ii) Geometry may sometimes provide us with first integrals. When a system of ODEs exhibits (continuous) symmetries, there are associated first integrals. This is actually a theorem due to E. Noether.

The two body problem: Let us consider a system consisting of two bodies namely the *sun* and a *planet*. In $\mathbb{R}^3$ the system is governed by Newton's laws of motion which is a system of six second order ODEs for the instantaneous positions of the *sun* and the *planet*. Equivalently a system of 12 first order ODEs three coordinates of position and three for velocity for each of the two objects namely the *sun* and the *planet*.

So we would need a set of 11 first integrals. First since there are no external forces, linear momentum is conserved. Denoting by $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities and m_1, m_2 the masses of the *sun* and the *planet* we have that

$$\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \mathbf{c} = \text{const}$$

which gives three first integrals and these in turn mean

$$\frac{d}{dt}\Big(\frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}\Big) = \mathbf{c}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of the *sun* and a *planet*.

We get

$$\frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} = \mathbf{c}t + \mathbf{d}$$

which gives three more first integrals. **The center of mass moves along a straight line.** Choosing the origin to be the center of mass we can decouple the system into two (practically identical) systems each with six equations. We need now five more first integrals. Law of conservation of angular momentum provides three more first integrals and law of conservation of energy provides one more totalling 10 first integrals.

Since the divergence of f in this case is zero a classical result of *Jacobi* asserts that we can find one more first integral ! and so we are exactly in the situation where we have the solution curve *but not its parametrization!*

Finding the parametrization leads to the problem of inverting the Kepler's equation that we now turn to. To derive the Kepler equation we shall use the Kepler's laws which are really the result of these first integrals. The second law for instance is the law of conservation of angular momentum and the third law arises out of the energy equation.