Fourier Analysis and its Applications

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48 Compact self-adjoint operators. Existence of eigen-values contd...

Invariant ortho-complements: Suppose that H is a Hilbert space, $T : H \longrightarrow H$ is a self adjoint bounded linear operator on H. We say that Y is a T-invariant subspace if it is a vector subspace and

$$Tv \in Y, \qquad \text{whenever } v \in Y.$$

Let us now show that if Y is a T-invariant subspace of H then $Y^{\perp}$ is also T-invariant. Well, suppose $z \in Y^{\perp}$ we have to show $Tz \in Y^{\perp}$ namely

$$\langle Tz, y \rangle = 0, \qquad \text{for all } y \in Y.$$

By self-adjointness, LHS equals $\langle z, Ty \rangle$ and $Ty \in Y$, $z \in Y^{\perp}$ implies

$$\langle z, Ty \rangle = 0$$

proving the claim.

Restricting T to $Y^{\perp}$ we get an operator

$$T : Y^{\perp} \longrightarrow Y^{\perp}.$$

which is evidently a bounded linear operator and $Y^{\perp}$ being closed is itself a Hilbert space in its own right. The equation

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

holds for all $x, y \in Y^{\perp}$ and so the restriction of T to $Y^{\perp}$ is also a self-adjoint operator. If T is a compact operator to begin with then so would this restriction.

Theorem (Existence of an orthonormal basis of eigen-vectors): Suppose $T : H \longrightarrow H$ is a compact self-adjoint operator on a Hilbert space then there is an orthonormal basis of eigen-vectors of H. Each non-zero eigen value of T has finite multiplicity.

Proof: We have already established the existence of an eigen-vector. A simple application of Zorn's lemma or Hausdorff's maximality theorem assures us that there is a maximal linearly independent orthonormal set B in H consisting of eigen-vectors and this B is non-empty.

Claim: B is a basis for H. Suppose not let Y be the closure of the linear span of B. B is not a basis means $Y \neq H$ and so $Y^{\perp}$ is not the zero space and it is a Hilbert space and the opertator restricted to $Y^{\perp}$ is also compact self-adjoint as observed and so it has an eigen vector v_0 which may be assumed to be a unit vector. Then v_0 is orthogonal to all the vectors in B and $B \cup \{v_0\}$ is a strictly larger orthonormal set of eigen-vectors of T contradicting the maximality of B. The claim is proved and therewith part (i) of the theorem.

Now let us show that each non-zero eigen-value has finite multiplicity. Let λ be a *NON-ZERO* eigen-value with eigen subspace V_{λ}. We have to show V_{λ} has finite dimension. Now let U_0 be the closed unit ball in V_{λ}.

If we show that U_0 is compact then we would be done. Well, Let $\{v_n\}$ be a sequence of vectors in U_0 then $Tv_n = \lambda v_n$ but Tv_n has a convergent subsequence whereby v_n itself must have a convergent subsequence which shows that U_0 is compact.

The same argument shows that for each $c > 0$ there are only finitely many eigen values λ with $|\lambda| > c$. Suppose not then we can select an infinite sequence $\{v_n\}$ of unit eigen vectors with pairwise distinct eigen-values forming a *monotone sequence*. The eigen-vectors have mutual distance $\sqrt{2}$ from each other. But then the image vectors $Tv_n = \lambda_n v_n$ have a convergent subsequence Tv_{n_k} and then

$$v_{n_k} = \lambda_{n_k}^{-1} T v_{n_k}$$

itself must converge in *norm* since $|\lambda_{n_k}^{-1}| < c^{-1}$ and so we again get a contradiction.

Distribution of eigen-values: So we have established that for a compact self-adjoint operator on a Hilbert space there is an orthonormal basis of eigen-vectors. Each non-zero eigen value has finite multiplicity.

The eigen values form a countable set and can only accumulate at zero.

We have also seen that the eigen-values can be generated by successively maximizing the Rayleigh quotients analogous to the case of a real symmetric matrix.

Weighted L^2 spaces Let $\rho : [0,1] \longrightarrow \mathbb{R}$ be *non-negative* continuous, whose zeros form a set of measure zero. Then $L^2_\rho[0,1]$ denotes the set of all $f : [0,1] \longrightarrow \mathbb{R}$ such that

$$\int_0^1 |f(t)|^2 \rho(t) dt < \infty.$$

This is a vector space over the reals on which we can define an inner product

$$\langle f, g \rangle = \int_0^1 f(t) g(t) \rho(t) dt.$$

With this $L^2_\rho[0,1]$ is a Hilbert space. Proof is the same as the usual $L^2[0,1]$ with cosmetic changes. The space is the correct set up for the study of the Sturm Liouville problem

$$y'' + \lambda \rho y = 0, \quad y(0) = y(1) = 0.$$

Sturm Liouville problems revisited We make the observation that the operator $T : L^2_\rho[0,1] \longrightarrow L^2_\rho[0,1]$ given by

$$Tf(x) = \int_0^1 K(x,t) f(t) \rho(t) dt \tag{7.39}$$

is a compact operator when $K(x,t)$ is continuous on $[0,1] \times [0,1]$ and is self-adjoint precisely when the kernel $K(x,t)$ is symmetric namely $K(x,t) = K(t,x)$.

Since the proofs parallel the classical case of $L^2[0,1]$ we shall leave these for the audience. We now tranform the Sturm Liouville problem into an eigen-value problem for a compact self-adjoint operator of the form (7.39). Let $y(x)$ be an eigen function of

$$y'' + \lambda \rho y = 0, \quad y(0) = y(1) = 0. \tag{7.40}$$

with eigen-value λ. We know that 0 is not an eigen value and so $\lambda \neq 0$. Integrating (7.40) twice in succession we get

$$\begin{aligned} y'(x) &= y'(0) - \lambda \int_0^x y(t) \rho(t) dt \\ y(x) &= xy'(0) - \lambda \int_0^x ds \int_0^s y(t) \rho(t) dt \end{aligned}$$

Switching the order of integration we get

$$y(x) = xy'(0) - \lambda \int_0^x y(t)\rho(t)dt \int_t^x ds$$

and so finally

$$y(x) = xy'(0) - \lambda \int_0^x (x-t)y(t)\rho(t)dt$$

The boundary condition $y(1) = 0$ now gives

$$y'(0) = \lambda \int_0^1 (1-t)y(t)\rho(t)dt$$

Substituting into the previous formula gives

$$\begin{aligned} y(x) &= \lambda \int_0^1 x(1-t)y(t)\rho(t)dt - \lambda \int_0^x (x-t)y(t)\rho(t)dt \\ &= \lambda \int_0^1 G(x,t)y(t)\rho(t)dt \end{aligned} \tag{7.41}$$

The Green's function $G(x,t)$ is given by

$$\begin{aligned} G(x,t) &= t(1-x) \text{ if } t \le x \\ G(x,t) &= x(1-t) \text{ if } x \le t. \end{aligned}$$

We see that the Green's function is symmetric and λ^{-1} is an eigen-value of the compact self-adjoint operator $T : L^2_\rho[0,1] \longrightarrow L^2_\rho[0,1]$ given by

$$Tf(x) = \int_0^1 G(x,t)f(t)\rho(t)dt. \tag{7.42}$$

Conversely suppose $y(x)$ is an eigen-function for the operator (7.42) with eigen-value λ^{-1} namely (7.41) holds. We show that $y(x)$ is an eigen-function for the BVP $y'' + \lambda\rho y = 0$ with zero boundary conditions. Now differentiating the equation $Ty(x) = \lambda^{-1}y(x)$ (where $Ty(x)$ is given by (7.42)), we get

$$\begin{aligned} y'(x) &= \lambda \int_0^1 G'(x,t)y(t)\rho(t)dt \\ &= \lambda\Big(-\int_0^x ty(t)\rho(t)dt + \int_x^1 (1-t)y(t)\rho(t)dt \Big). \\ y'' &= \lambda(-xy(x)\rho(x) - (1-x)y(x)\rho(x)) \\ y'' &= -\lambda\rho(x)y(x) \end{aligned}$$

Now directly from the expression for $G(x,t)$ we see that $G(x,t) = 0$ when $x = 0$ and $x = 1$ so that $y(x)$ also satisfies the boundary conditions.

Theorem: The Sturm Liouville problem

$$y'' + \lambda \rho y = 0, \quad y(0) = 0 = y(1)$$

has a countable set of eigen values tending to infinity. Each eigen-value is simple and the eigen-functions form a complete orthonormal basis for $L^2_\rho[0,1]$. In other words every $f \in L^2_\rho[0,1]$ can be written as a Fourier series

$$f = \sum_{n=1}^{\infty} c_n y_n$$

where y_n are the eigen-functions of the Sturm-Liouville problem.