Fourier Analysis and its Applications

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47 Compact self-adjoint operators. Existence of eigen-values

We see that

$$\begin{aligned} 2\|Y\|^2 &\le \sup_n(\|v_{n,n}\|^2) + \|v_{n,n} - 2(a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N)\|^2 \\ &\le 2\sup_n(\|v_{n,n}\|^2) + 4\|a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N\|^2 - 4\langle v_{n,n}, a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N\rangle \end{aligned}$$

Letting $n \to \infty$ we get that

$$\|a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N\|^2 = \|Y\|^2 \le \sup_n(\|v_{n,n}\|^2)$$

So we get for some constant A

$$|a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots + |a_N|^2 < A, \quad \text{for all } N \in \mathbb{N}.$$

Proof of the theorem is complete.

Remarks on the Banach Alaoglu theorem for separable Hilbert spaces: The proof of the second part was long since it was elementary in character. If we permit ourselves the use of more sophisticated tools from functional analysis the proof would be considerably shorter. The relevant result would be the The Banach Alouglu theorem. See for instance Goffman-Pedrick (p. 210) though the result is not a named there. Also it must be noted that since the closed unit ball in an infinite dimensional space is NOT compact, norm boundedness of a sequence does not imply norm convergence of a subsequence but we have a *weak substitute* namely *a weakly converging subsequence*!

We have seen that there is NO HOPE of getting norm convergence out of weakly converging sequences. However there is one important situation that under a *peculiar* additional condition we can recover norm-convergence out of a weakly convergent sequence. This result is often useful but since we shall not make use of it we shall not prove it here

Theorem: Suppose H is a Hilbert space and $\{v_n\}$ is a sequence that converges weakly to v. Assume that

$$\|v_n\| \longrightarrow \|v\|$$

then, we have $v_n \longrightarrow v$ in norm.

Exercise: Prove this using the parallalogram identity.

We are now ready for the proof of the spectral theorem for a compact self-adjoint operator on a Hilbert space. As in the case of matrices we have the following results:

Theorem (spectral theorem): Suppose $T : H \longrightarrow H$ is a self-adjoint operator on a Hilbert space,

(i) $\langle Tv, v\rangle$ is real for all $v \in H$.

(ii) Eigen-values of T (if they exist) are real.

(iii) The eigen-vectors corresponding to distinct eigen-values are mutually perpendicular.

Proof: The first two results are significant only for complex Hilbert spaces. In our proof of the spectral theorem we will directly produce a real eigen-value by construction and so the proofs of (i) and (ii) would be subsumed in the proof of the spectral theorem. Nevertheless we give an elementary proof in the complex case.

Well,

$$\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$$

The result is now clear.

Turning to (ii) suppose λ is an eigen-value of T with eigenvector v. Then

$$\langle Tv, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2$$

On the other hand

$$\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle = \overline{\lambda}\|v\|^2$$

comparing the two we get the result. Finally suppose v and w are eigen-vectors corresponding to distinct eigen-values λ and μ. Then $\langle Tv, w\rangle = \langle v, Tw\rangle$ translates to

$$\langle \lambda v, w\rangle = \langle v, \mu w\rangle$$

Which means (in view of (ii)) $(\lambda - \mu)\langle v, w\rangle = 0$. We conclude $\langle v, w\rangle = 0$.

Existence of eigen-values Recall that if A is a $n \times n$ real symmetric matrix then

$$\sup\{\langle Av, v\rangle \ : \ \|v\| = 1\}$$

is attained at an eigen-value of A. We proved this in chapter 6. But the compactness of the closed unit ball in $\mathbb{R}^n$ was crucial. We do not have this compactness property in infinite dimensional spaces and we must somehow resort to the *weak substitute* we have established namely the existence of a weakly convergent subsequence. This is the crux of the proof of the spectral theorem. The rest is routine !

The fact of the matter is that the same idea of maximizing the Rayleigh quotient

$$\langle Tv, v\rangle$$

works in the infinite dimensional setting with the careful use of weak compactness of the unit ball which is why the Banach-Alaoglu theorem is so important.

Existence of eigen-values: Before embarking upon the theorem let us make a simple observation (we consider only real Hilbert spaces). Now suppose $\langle Tv, v\rangle = 0$ for all $v \in H$ then $T = 0$. To see this observe that

$$\begin{aligned}\langle T(v+w), v+w\rangle &= 0\\ \langle T(v-w), v-w\rangle &= 0\end{aligned}$$

Subtracting we get

$$\langle Tv, w\rangle + \langle Tw, v\rangle = 0$$

which implies $\langle Tv, w\rangle = 0$ for all $v, w \in H$. Taking $w = Tv$ we conclude that $T \equiv 0$. We may clearly assume that T is not the zero operator and thus the Rayleigh quotient cannot be identically zero. Replacing T by $-T$ we may assume that the Rayleigh quotient assumes some strictly positive values and so *its supremum is positive.*

Theorem: Suppose $T : H \longrightarrow H$ is a compact self-adjoint operator on a Hilbert space then

$$\sup\{\langle Tv, v\rangle \; : \; \|v\| = 1\}$$

is an eigen-value of T and is attained at an eigen-vector of H.

Proof: Denote the supremum by λ which evidently exists and it is positive by the remarks made at the beginning. Take a sequence $\{v_n\}$ of *unit vectors* such that

$$\langle Tv_n, v_n\rangle \longrightarrow \lambda.$$

Since T is a compact operator, the image sequence Tv_n has a norm-convergent subsequence and we work with this subsequence in what follows and continue to denote it by v_n. Thus we have that Tv_n converges to say y.

Further the sequence $\{v_n\}$ being norm-bounded, we have a weakly convergent subsequence and we work with this subsequence denoting it by $\{v_n\}$. Thus we have a v_0 such that for every $w \in H$,

$$\langle v_n - v_0, w\rangle \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

Now

$$\langle Tv_n, v_n\rangle = \langle Tv_n - y, v_n\rangle + \langle y, v_n\rangle \longrightarrow \langle y, v_0\rangle \tag{7.36}$$

Thus $\lambda = \langle y, v_0\rangle$. Observe that since $\langle Tv_n, v_n\rangle$ is real, the limit λ is also real. now by self-adjointness of T we have

$$\langle Tv_n, z\rangle = \langle v_n, Tz\rangle \longrightarrow \langle v_0, Tz\rangle = \langle Tv_0, z\rangle \tag{7.37}$$

and at the same time (by norm convergence of Tv_n),

$$\langle Tv_n, z\rangle \longrightarrow \langle y, z\rangle \tag{7.38}$$

comparing (7.37) and (7.38) we get that

$$Tv_0 = y.$$

Going back to (7.36), we get

$$\lambda = \langle y, v_0\rangle = \langle Tv_0, v_0\rangle$$

Note that since $\lambda > 0$ the vector v_0 cannot be the zero vector. Let us show that it is a unit vector. First, for a fixed m,

$$\langle v_n, v_m\rangle \longrightarrow \langle v_0, v_m\rangle$$

we see that

$$|\langle v_0, v_m\rangle| \leq 1.$$

Letting $m \to \infty$ we get that $\|v_0\| \leq 1$. On the other hand,

$$\langle T\Big(\frac{v_0}{\|v_0\|}\Big), \Big(\frac{v_0}{\|v_0\|}\Big)\rangle \leq \lambda$$

Which means

$$\lambda = \langle Tv_0, v_0\rangle \leq \lambda\|v_0\|^2$$

implying that $\|v_0\| = 1$.

We have finally shown that the supremum of the Rayleigh quotients IS attained at a unit vector v_0. The most difficult part of the proof is over. We show that v_0 is an eigen-vector corresponding to eigen-value λ. We can imitate the finite dimensional case here. Let $h \in H$ be arbitrary and $t \in \mathbb{R}$ be small in absolute value. Then

$$\langle \frac{T(v_0 + th)}{\|v_0 + th\|}, \frac{v_0 + th}{\|v_0 + th\|} \rangle \leq \lambda$$

clearing the denominators we get

$$\langle T(v_0 + th), v_0 + th \rangle \leq \lambda \|v_0 + th\|^2$$

Expanding, and using $\langle v_0, v_0 \rangle = \lambda$ and $\|v_0\| = 1$ we get

$$2t\langle Tv_0, h \rangle \leq 2t\lambda \langle v_0, h \rangle + t^2(\dots)$$

Dividing by $|t|$ and letting $t \to 0$ through positive values and through negative values we get

$$\langle Tv_0 - \lambda v_0, h \rangle = 0$$

Since $h \in H$ was arbitrary we conclude that

$$Tv_0 = \lambda v_0.$$

Thus v_0 is an eigen-vector. In particular eigen-values and eigen-vectors exist.