Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

46 The Banach Alaoglu theorem

Proof of (iii) is more involved. Suppose $\lambda \neq 0$ then we show that $T - \lambda I$ is surjective and hence invertible since injectivity is already established in (ii). Given $g \in L^2[0,1]$ we need to solve

$$\int_0^x f(t)dt - \lambda f(x) = g(x) \tag{7.28}$$

for $f \in L^2[0,1]$. To begin with let us solve (7.28) in case g is smooth. We shall obtain a formula for f and observe that the formula also provides a solution when g is not smooth ! Differentiating (7.28) we get the ODE

$$f(x) - \lambda f'(x) = g'(x).$$

We solve this ODE for f using the method of variation of parameters.

The solution is given by

$$f(x) = ce^{x/\lambda} - \frac{1}{\lambda}\int_0^x g'(t)e^{(x-t)/\lambda}dt.$$

where c is an arbitrary constant whose value would be fixed presently. Integrating by parts we get

$$f(x) = (c + g(0)/\lambda)e^{x/\lambda} - \frac{g(x)}{\lambda} - \frac{1}{\lambda^2}\int_0^x g(t)e^{(x-t)/\lambda}dt.$$

If we select $c = -g(0)/\lambda$ we get

$$f(x) = -\frac{g(x)}{\lambda} - \frac{1}{\lambda^2}\int_0^x g(t)e^{(x-t)/\lambda}dt. \tag{7.29}$$

Now that we have an explicit formula for the solution of the equation $Tf - \lambda f = g$ we see that (7.29) makes *perfect sense even if* g *is not smooth* but merely in $L^2[0,1]$.

But now we must go back and verify that (7.29) is a valid solution of

$$Tf - \lambda f = g$$

when $g \in L^2[0,1]$. Integrating (7.29) we get

$$Tf(y) = -\frac{1}{\lambda}\int_0^y g(x)dx - \frac{1}{\lambda^2}\int_0^y dx \int_0^x g(t)e^{(x-t)/\lambda}dt$$

Switching the order of integral we get

$$Tf(y) = -\frac{1}{\lambda}\int_0^y g(x)dx - \frac{1}{\lambda^2}\int_0^y g(t)dt \int_t^y e^{(x-t)/\lambda}dx$$

Evaluating the inner integral,

$$Tf(y) = -\frac{1}{\lambda}\int_0^y g(t)e^{(y-t)/\lambda}dt$$

We see at once (using (7.29)) that

$$Tf(y) - \lambda f(y) = g(y)$$

and the verification is complete and $T - \lambda I$ is surjective if $\lambda \neq 0$.

It is a general fact that the spectrum of any bounded operator on a Hilbert space is always non-empty but the Volterra operator shows that it can reduce to a singleton !

We shall see that this pathology does not occur when the compact operator is also self-adjoint. For a *compact self-adjoint* operator on a Hilbert space we have a basis of eigen-vectors and the spectral theorem holds. However we shall need a preliminary result before we embark upon the spectral theorem.

Weak convergence of a sequence Suppose $\{v_n\}$ is a sequence of vectors in a Hilbert space then recall that v_n is said to converge to v in norm if $\|v_n - v\| \longrightarrow 0$ when $n \longrightarrow \infty$. We now introduce a *much weaker* notion

Definition: A sequence $\{v_n\}$ of vectors in a Hilbert space H is said to converge WEAKLY to v if for EVERY $w \in H$,

$$\langle v_n - v, w \rangle \longrightarrow 0$$

when $n \longrightarrow \infty$.

We say v is the weak limit of the sequence $\{v_n\}$.

Using the Cauchy Schwartz inequality we see at once that convergence in norm implies weak convergence. The converse is not true.

Example: Consider $L^2[-\pi, \pi]$ and the sequences $\{\cos nx\}$ and $\{\sin nx\}$. These converge to zero weakly by Riemann Lebesgue Lemma. In fact *any* orthonormal sequence $\{\mathbf{b}_n\}$ in a separable Hilbert space *converges weakly to zero*. In fact we shall enlarge the sequence to form an orthonormal basis and show that the enlarged sequence goes to zero weakly.

Let $v \in H$ and x_j be the j-th Fourier coefficient of v with respect to $\{\mathbf{b}_n\}$. Parseval formula gives

$$\|v\|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + \dots$$

which means $x_n \longrightarrow 0$ or

$$\langle \mathbf{b}_n, v \rangle \longrightarrow 0$$

for every $v \in H$ and the claim is established.

An orthonormal sequence is FAR from being a norm convergent sequence since

$$\|\mathbf{b}_n - \mathbf{b}_m\| = \sqrt{2}, \quad \text{for all } m \neq n.$$

Theorem (Weak and norm convergence): (i) Suppose $\{v_n\}$ is a sequence of vectors in a Hilbert space converging weakly then the sequence is norm bounded namely there is a constant M such that $\|v_n\| \leq M$ for all $n \in \mathbb{N}$.

(ii) Suppose $\{v_n\}$ is a sequence of vectors in a Hilbert space which is norm bounded then there is a *subsequence* converging weakly.

Proof: Proof of (i) is a nice application of the Banach-Steinhaus's theorem. For each n, let $T_n : H \longrightarrow \mathbb{C}$ be the linear transformation given by

$$T_n(w) = \langle w, v_n \rangle$$

It is clear that T_n is a bounded linear map since $|T_n w| \leq \|v_n\|\|w\|$. Since $T_n(w)$ converges for each w we see that the sequence $\{T_n w\}$ is bounded for each $w \in H$ by a bound M_w that may depend on w. The Banach Steinhaus theorem says that there is an M such that

$$|T_n w| \leq M, \quad \text{for all } n \in \mathbb{N} \quad \text{and for all } w \in U$$

where U is the closed unit ball in H. In particular for all unit vectors w,

$$|\langle w, v_n \rangle| \leq M, \quad \text{for all } n \in \mathbb{N}.$$

Taking $w = v_n/\|v_n\|$ we see that $\|v_n\| \leq M$ for all $n \in \mathbb{N}$ and the proof of (i) is complete.

We shall prove (ii) under the assumption that H is separable. Take a countable orthonormal basis for H say:

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots\}$$

Since $\{v_n\}$ is norm bounded, the sequence $\langle v_n, \mathbf{b}_1\rangle$ is bounded and so there is a convergent subsequence namely,

$$\langle v_{n_1}, \mathbf{b}_1\rangle \longrightarrow a_1.$$

It is convenient to denote the subsequence $\{v_{n_1}\}$ by

$$v_{1,1}, v_{1,2}, v_{1,3}, \dots. \tag{7.30}$$

Now since $\{v_{1,n}\}$ is norm bounded, the sequence $\langle v_{1,n}, \mathbf{b}_2\rangle$ is bounded and so there is a convergent subsequence namely,

$$\langle v_{1,n_j}, \mathbf{b}_1\rangle \longrightarrow a_2.$$

It is convenient to denote the subsequence $\{v_{1,n_j}\}$ by

$$v_{2,1}, v_{2,2}, v_{2,3}, \dots. \tag{7.31}$$

Proceeding thus we get a family of sequences *with each one being a subsequence of the preceeding.* Then the diagonal sequence

$$v_{1,1}, v_{2,2}, v_{3,3}, \dots. \tag{7.32}$$

has the property that

$$\langle v_{n,n}, \mathbf{b}_j\rangle$$

converges for each $j = 1, 2, 3, \dots$.

Now we make an educated guess as to what is the weak limit of $\{v_{n,n}\}$ and then we shall verify that our guess is correct. The weak limit has a Fourier expansion

$$w = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + \dots$$

and $\langle v_{n,n}, \mathbf{b}_j\rangle \longrightarrow \langle w, \mathbf{b}_j\rangle$ implies that $x_j = a_j$ for all j and so our guess for the weak limit is

$$w = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + a_3\mathbf{b}_3 + \dots \tag{7.33}$$

We have the serious job of checking that the series (7.33) converges in H which will be so if and only if

$$|a_1|^2 + |a_2|^2 + |a_3|^2 + \dots < \infty \tag{7.34}$$

We shall verify (7.34) later and continue with the argument. We have to show that for each $v \in H$ there holds $\langle v_{n,n} - w, v\rangle \longrightarrow 0$. which clearly holds for $v = \mathbf{b}_j$ for every j and hence holds for every element p in the *linear span of* B. Let $v \in H$ and $\epsilon > 0$ be arbitrary. There is an element p in the *linear span of* B such that

$$\|v - p\| < \epsilon/2M$$

where M exceeds $\|v_n\| + \|w\|$. Also for this p there exists $n_0 \in \mathbb{N}$ such that

$$\langle v_{n,n} - w, p\rangle < \epsilon/2, \qquad \text{for all } n > n_0.$$

Then for $n > n_0$ we have

$$\begin{aligned} |\langle v_{n,n} - w, v\rangle| &\leq |\langle v_{n,n} - w, p\rangle| + \|p - v\|(\|v_{n,n}\| + \|w\|) \\ &< \epsilon/2 + \epsilon/2 \end{aligned}$$

and that would complete the argument as soon as we prove the claim (7.34).

The proof of (7.34) is a nice application of the parallelogram law as we shall now see. First, we fix N and look at

$$\|v_{n,n} - 2(a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N)\|^2 \tag{7.35}$$

which expands to (remember our Hilbert spaces are real which suffices for our applications):

$$\|v_{n,n}\|^2 + 4\|a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N\|^2 - 4\langle v_{n,n}, a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N\rangle$$

The last two terms (together) tend to zero as $n \to \infty$ (Easy).

Now let us apply the parallalogram identity with

$$X = v_{n,n} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N), \quad Y = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N$$

and we get

$$\|X + Y\|^2 + \|X - Y\|^2 = 2(\|X\|^2 + \|Y\|^2)$$

which means

$$2\|Y\|^2 \leq \|v_{n,n}\|^2 + \|v_{n,n} - 2(a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N)\|^2$$

We see that

$$\begin{aligned} 2\|Y\|^2 &\leq \sup_n(\|v_{n,n}\|^2) + \|v_{n,n} - 2(a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N)\|^2 \\ &\leq 2\sup_n(\|v_{n,n}\|^2) + 4\|a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N\|^2 - 4\langle v_{n,n}, a_1\mathbf{b}_1 + \cdots + a_N\mathbf{b}_N\rangle \end{aligned}$$

Letting $n \to \infty$ we get that

$$\|a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_N\mathbf{b}_N\|^2 = \|Y\|^2 \leq \sup_n(\|v_{n,n}\|^2)$$

So we get for some constant A

$$|a_1|^2 + |a_2|^2 + |a_3|^2 + \cdots + |a_N|^2 < A, \quad \text{for all } N \in \mathbb{N}.$$

Proof of the theorem is complete.

Remarks on the Banach Alaoglu theorem for separable Hilbert spaces: The proof of the second part was long since it was elementary in character. If we permit ourselves the use of more sophisticated tools from functional analysis the proof would be considerably shorter. The relevant result would be the The Banach Alouglu theorem. See for instance Goffman-Pedrick (p. 210) though the result is not a named there. Also it must be noted that since the closed unit ball in an infinite dimensional space is NOT compact, norm boundedness of a sequence does not imply norm convergence of a subsequence but we have a *weak substitute* namely *a weakly converging subsequence*!