Fourier Analysis and its Applications

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45 Spectrum of a bounded operator

Self-adjoint operators on a Hilbert space Recall that if A is an $n \times n$ real symmetric matrix then

$$\langle Ax, y\rangle = \langle x, Ay\rangle, \qquad \text{for all } x, y \in \mathbb{R}^n$$

where $\langle x, y\rangle$ denotes the usual dot product in $\mathbb{R}^n$.

The definition of self-adjoint operators on a Hilbert space is modelled on this.

A *bounded* operator $T : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad \text{for all } x, y \in H.$$

Let us now examine when is a Hilbert-Schmidt operator with a continuous kernel self-adjoint? We shall deal with real valued kernels only. If the kernel is complex valued one has to modify the result by putting the complex conjugation at appropriate places ! Our concern here is solution operators of two point boundary value problems with continuous *real* density $\rho(x)$ which is positive almost everywhere.

Theorem (Symmetric kernels and Self-adjoint operators): Suppose $K(x, t)$ is continuous on $[0, 1] \times [0, 1]$ then the *Hilbert-Schmidt operator* $T : L^2[0, 1] \longrightarrow L^2[0, 1]$ given by

$$Tf(x) = \int_0^1 K(x, t)f(t)dt$$

is self-adjoint if and only if the kernel is symmetric.

In particular for our boundary value problem with Dirichlet boundary conditions at 0 and 1 the solution operator is a compact self-adjoint operator. On the other hand *the Volterra operator is NOT a self-adjoint operator*

Proof of the theorem We shall assume that we are working with a real Hilbert space. Let $f, g \in L^2[0, 1]$. Then

$$\begin{aligned} \langle Tf, g\rangle &= \int_0^1 \int_0^1 K(x, t)f(t)g(x)dtdx \\ \langle f, Tg\rangle &= \int_0^1 \int_0^1 K(t, x)f(t)g(x)dxdt \end{aligned}$$

So that the condition $\langle Tf, g\rangle = \langle f, Tg\rangle$ translates to

$$\int_0^1 \int_0^1 (K(x, t) - K(t, x))f(t)g(x)dxdt = 0, \qquad \text{for all } f, g \in L^2[0, 1].$$

This implies (using *Weierstrass's approximation theorem* for instance) that

$$K(x, t) = K(t, x)$$

Eigen-values: Recall that in linear algebra if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map then the following are equivalent:

(i) T is injective.

(ii) T is surjective.

In infinite dimensional spaces the above equivalence FAILS.

Let H be a Hilbert space and $T : H \longrightarrow H$ be a continuous linear map. A complex number λ is said to be in the spectrum of T if at least one of the following holds:

(i) $T - \lambda I$ is fails to be injective.

(ii) $T - \lambda I$ is fails to be surjective.

We we are mainly concerned with the failure of (i) and accordingly we say λ is an *eigen-value* of T if $T - \lambda I$ fails to be injective. In other words, λ is said to be an eigen-value of T if there is a non-zero vector $v \in H$ such that $Tv = \lambda v$.

The Open Mapping Theorem Suppose X and Y are metric spaces and $f : X \longrightarrow Y$. Then f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y. One can ofcourse replace every occurrence of the word open by the word closed.
Suppose $f : X \longrightarrow Y$ is a bijective continuous map. When can we say $f^{-1} : Y \longrightarrow X$ is continuous? This is an important question that leads to the notion of open/closed map. The requirement is that for each O open in X we need $(f^{-1})^{-1}(O)$ to be open in Y namely $f(O)$ should be open in Y whenever O is open in X. Again we could replace the word open by closed and we would again get a condition for f^{-1} to be continuous.

Definition: A map $f : X \longrightarrow Y$ is said to be an open (closed) map if whenever O is open (resp. closed) in X the set $f(O)$ is open (resp. closed) in Y.

Thus a bijective continuous map $f : X \longrightarrow Y$ is a homeomorphism if it is an open (closed) map.

(i) Suppose X is a compact metric space and Y is a metric space then a continuous function $f : X \longrightarrow Y$ is also a closed map and hence a continuous bijection from X onto Y is a homeomorphism.

(ii) Suppose G is a connected open set in $\mathbb{C}$ and $f : G \longrightarrow \mathbb{C}$ is a non constant holomorphic function, then f is an open mapping.

(iii) Suppose G is a connected open set in $\mathbb{R}^n$ and $f : G \longrightarrow \mathbb{R}^n$ is smooth and $Df(x)$ is non-singular at each point of G then f is an open mapping.

Theorem: Let H be a Hilbert space and $T : H \longrightarrow H$ be a *surjective* continuous linear map. Then T is an open mapping.

Volterra operator revisited: Let us examine the Volterra operator again for spectral values.

$$Tf(x) = \int_0^x f(t)dt.$$

First of all note that since the image of the Volterra operator contains only continuous functions it is not surjective and so 0 is in the spectrum.

Questions: (i) Is 0 an eigen-value? What are the eigen-values of the Volterra operator?

(ii) We shall see that the Volterra operator has NO eigen-values.

(iii) Further we shall show that the *spectrum of the Volterra operator consists of the zero element alone.*

In fact if H is an infinite dimensional Hilbert space and $T : H \longrightarrow H$ is a compact operator then 0 is always in the spectrum of T. The Volterra operator exhibits an extreme phenomenon.

Spectrum of the Volterra operator (i) Let us first show that 0 is not an eigen-value. Suppose it is. Then there exists a non-zero L^2 function f such that

$$\int_0^x f(t)dt = 0, \qquad \text{for all } x \in [0,1].$$

This means

$$\int_x^y f(t)dt = 0, \qquad \text{for all } x, y \in [0,1] \text{ with } x < y.$$

Hence

$$\int_A f(t)dt = 0$$

for all measurable subsets of $[0,1]$. This forces $f \equiv 0$ which is a contradiction.

We have proved 0 is not an eigen-value.

(ii) Let $\lambda \neq 0$ be a complex number. Let us now show that λ cannot be an eigen-value. Suppose it is and let $f \in L^2[0,1]$ be the eigen-vector namely $Tf = \lambda f$ or

$$\int_0^x f(t)dt = \lambda f(x). \tag{7.27}$$

Since the LHS is continuous, so is the RHS which means that the eigen-vector f had to be continuous to begin with. Now the LHS is continuously differentiable and so is the RHS so that the eigen-vector f had to be continuously differentiable to begin with. Proceeding thus we see that the eigen-vector must be differentiable infinitely often. Differentiating the displayed equation (7.27) we get

$$f(x) = \lambda f'(x)$$

which means $f(x) = ce^{x/\lambda}$. With this we see that (7.27) fails ($c \neq 0$).

Proof of (iii) is more involved. Suppose $\lambda \neq 0$ then we show that $T - \lambda I$ is surjective and hence invertible since injectivity is already established in (ii). Given $g \in L^2[0,1]$ we need to solve

$$\int_0^x f(t)dt - \lambda f(x) = g(x) \tag{7.28}$$

for $f \in L^2[0,1]$. To begin with let us solve (7.28) in case g is smooth. We shall obtain a formula for f and observe that the formula also provides a solution when g is not smooth ! Differentiating (7.28) we get the ODE

$$f(x) - \lambda f'(x) = g'(x).$$

We solve this ODE for f using the method of variation of parameters.