Fourier Analysis and its Applications

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44 Hilbert-Schmidt operators. Green's functions

More examples of compact operators: The example can easily be generalized to other integral operators. Again let us work with $L^2[0,1]$ for convenience.

Example 1 (Hilbert-Schmidt operator): $T : L^2[0,1] \longrightarrow L^2[0,1]$ given by

$$Tf(x) = \int_0^1 K(x,t)f(t)dt.$$

where $K(x,t)$ is a continuous function on $[0,1] \times [0,1]$. Again let us show that this operator is a compact operator.

We shall first establish that the family $\{Tf \ : \ f \in U\}$ is equi-continuous. This is easy. Since every continuous function on $[0,1] \times [0,1]$ is uniformly continuous, given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - y| < \delta, \qquad \text{implies } |K(x,t) - K(y,t)| < \epsilon.$$

The Hilbert-Schmidt Operator: Hence, for $|x - y| < \delta$ we have

$$|Tf(x) - Tf(y)| \leq \int_0^1 |K(x,t) - K(y,t)||f(t)|dt < \epsilon \int_0^1 |f(t)|dt < \epsilon$$

since the integral of $|f|$ over $[0,1]$ is less than or equal to $\|f\|$ by Cauchy Schwartz and $\|f\| \leq 1$.

To establish uniform boundedness we proceed as follows. Let M be the supremum of $K(x,t)$ over the square $[0,1] \times [0,1]$. Then

$$|Tf(x)| \leq \int_0^1 |K(x,t)||f(t)|dt \leq M \int_0^1 |f(t)|dt \leq M\|f\| \leq M.$$

The Ascoli-Arzela theorem now proves that the operator is compact.

Connection with boundary value problems Let us see how Hilbert Schmidt operators arise out of boundary value problems. Consider the problem of solving

$$y'' + \lambda\rho(x)y = f(x), \quad y(0) = y(1) = 0. \tag{7.23}$$

Let us assume that λ is NOT an eigen-value so that the problem with zero RHS (namely the corresponding homogeneous system) has *only the trivial solution*. This ensures that the problem at hand (if it has a solution) can only have a *unique* solution.

Compare the situation with solving $Ax = b$ where A and b are matrices. If the homogeneous equation $Ax = 0$ has only the trivial solution then the inhomogeneous equation $Ax = b$ cannot have at most one solution.

So if (7.23) has a solution for each $f \in L^2[0,1]$ then we obtain a well-defined solution operator $T : L^2[0,1] \longrightarrow L^2[0,1]$ namely, the unique solution to (7.23) for the given *input function* f. Let us see what is the form of this solution operator.

Form of the solution operator: Let $y_1(x)$ and $y_2(x)$ be the solutions of the initial value problem

$$y'' + \lambda\rho(x)y = 0, \tag{7.24}$$

satisfying

$$y_1(0) = 1, y_1'(0) = 0, \quad y_2(0) = 0, y_2'(0) = 1. \tag{7.25}$$

These two solutions are evidently linearly independent. We use the method of *variation of parameters* to solve (7.24). To this end we seek a particular solution $y_p(x)$ of the inhomogeneous equation (7.23) in the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{7.26}$$

where the functions $v_1(x)$ and $v_2(x)$ satisfy the pair of equations

$$v_1'y_1 + v_2'y_2 = 0, \quad v_1'y_1' + v_2'y_2' = f$$

Solving these we get

$$v_1(x) = -\int_0^x y_2(t)f(t)dt, \quad v_2(x) = \int_0^x y_1(t)f(t)dt.$$

So we get the particular solution

$$y_p(\xi) = \int_0^\xi (y_1(t)y_2(\xi) - y_1(\xi)y_2(t))f(t)dt = \int_0^1 K(\xi,t)f(t)dt$$

where the kernel $K(\xi,t)$ is given by

$$K(\xi,t) = y_1(t)y_2(\xi) - y_1(\xi)y_2(t) \ (t \le \xi), \quad \text{and } K(\xi,t) = 0 \ (t \ge \xi)$$

Observe that the kernel is continuous on $[0,1] \times [0,1]$. We need to modify the solution to fit the boundary conditions.

Incorporating the boundary conditions: We seek the solution of our problem as

$$y(x) = c_1y_1(x) + c_2y_2(x) + \int_0^x K(x,t)f(t)dt.$$

where the constants c_1 and c_2 have to be appropriately selected. The condition $y(0) = 0$ gives immediately $c_1 = 0$. Setting $x = 1$ we get

$$c_2 = \frac{-1}{y_2(1)}\int_0^1 K(1,t)f(t)dt.$$

How can we be sure that $y_2(1) \neq 0$? *Because $y_2(1) = 0$ would immediately say that y_2 is an eigen-function with eigen-value λ but we have expressly assumed that λ is not an eigen-value.*

The Green's function: Putting the value of c_2 obtained in our formula we get

$$\begin{aligned} y(x) &= \frac{1}{y_2(1)}\int_0^1 (K(x,t)y_2(1) - K(1,t)y_2(x))f(t)dt \\ &= \int_0^1 G(x,t)f(t)dt. \end{aligned}$$

The kernel $G(x,t)$ is called the *Green's function* for the boundary value problem. It is evidently continuous. The solution of the problem has been expressed as a Hilbert-Schmidt operator on $L^2[0,1]$.

We now examine the kernel $G(x,t)$ in more detail by writing out the complete expression.

Symmetry of the Green's function

(i) If $t \leq x$ then

$$y_2(1)G(x,t) = y_2(x)y_2(t)y_1(1) - y_2(t)y_2(1)y_1(x).$$

(ii) If $x \leq t$ then

$$y_2(1)G(x,t) = y_2(x)y_2(t)y_1(1) - y_1(t)y_2(1)y_2(x).$$

Observe that the Green's function is *symmetric* namely

$$G(x,t) = G(t,x).$$

What is cause of this symmetry? This is related to the fact that the two point boundary value problem with Dirichlet boundary conditions gives rise to a self-adjoint operator.

Comment: Since there is a parameter λ in the differential equation, the Green's function would also depend on this parameter and so we must write $G(x,t,\lambda)$ for the Green's function.

Symmetry of the Green's function and self-adjointness You may recall that in the theory of Poisson's equation with Dirichlet boundary conditions,

$$\Delta u = f, \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0.$$

the solution can be expressed as

$$u(x) = \int_\Omega G(x,\xi) f(\xi) d\xi$$

and the Green's function is symmetric. This is again related to the self-adjointess of the problem.

Exercises: (1) Consult books on PDEs for the expression for the Green's function for a ball and verify that it is symmetric.

(2) Determine the Green's function for $y'' + \lambda y = f$ on $[0,1]$ with Dirichlet boundary conditions.

Green's function for a Ball in $\mathbb{R}^n$: The basic fact is that the Green's function is a correction to the so called fundamental solution which in $\mathbb{R}^n$ is the Newtonian potential. Thus

$$G(x,\xi) = \frac{a}{\|x-\xi\|^{n-2}} + C(x,\xi), \quad n \geq 3.$$

where a is a constant and the correction term $C(x,\xi)$ is harmonic on an open neighborhood of the closed ball B and hence smooth. So the cause of concern is the term involving $\|x-\xi\|^{2-n}$.

Exercise: Examine whether $G(x,\xi) \in L^2(B \times B)$. The answer may depend on the space dimension n.

Self-adjoint operators on a Hilbert space Recall that if A is an $n \times n$ real symmetric matrix then

$$\langle Ax, y\rangle = \langle x, Ay\rangle, \quad \text{for all } x, y \in \mathbb{R}^n$$

where $\langle x, y\rangle$ denotes the usual dot product in $\mathbb{R}^n$.

The definition of self-adjoint operators on a Hilbert space is modelled on this.

A *bounded* operator $T : H \longrightarrow H$ on a Hilbert space H is said to be self-adjoint if

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \qquad \text{for all } x, y \in H.$$

Let us now examine when is a Hilbert-Schmidt operator with a continuous kernel self-adjoint? We shall deal with real valued kernels only. If the kernel is complex valued one has to modify the result by putting the complex conjugation at appropriate places ! Our concern here is solution operators of two point boundary value problems with continuous *real* density $\rho(x)$ which is positive almost everywhere.