Fourier Analysis and its Applications

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43 A non-separable Hilbert space. Space of almost periodic functions

Non-separable Hilbert spaces. Almost periodic functions Do non-separable Hilbert spaces exist? If so, are they useful ? Specifically, do they arise *naturally* in other parts of analysis? We shall say a few words about this with some references. The most important example concerns the space of *almost periodic functions* on the real line. This topic is of paramount interest in *dynamical systems and differential equations.* The definition goes back to *Harold Bohr* (brother of the physicist *Niels Bohr*).

Definition: A continuous function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be almost periodic if for every $\epsilon > 0$ there exists $L_\epsilon > 0$ such that *every* interval of length L_ϵ contains a point c such that

$$|f(x+c) - f(x)| < \epsilon, \qquad \text{for all } x \in \mathbb{R}.$$

In particular, for each $\epsilon > 0$ there is an ϵ-approximate period c.

Note that if c is an ϵ-approximate period then $2c$ is only a 2ϵ-approximate period and so the set of ϵ-approximate periods will NOT be an additive subgroup of $\mathbb{R}$. So the clause that *EVERY interval of length L_ϵ contains an ϵ-approximate period* is really a condition that should be compared with say uniform continuity in elementary analysis.
It is completely trivial to see that a continuous periodic function is almost periodic (how?).

Theorem:

1. An almost periodic function is uniformly continuous.
2. The set AP of almost periodic functions on $\mathbb{R}$ is a vector space which is closed under pointwise multiplication and also closed under uniform limits.

In order to define an inner-product on AP we need to introduce a new notion.

DEfinition (Mean value of an almost periodic function):

1. Suppose $f \in AP$ then the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)dt$$

exists uniformly with respect to x and it is independent of x. This limit M_f is called the *Mean Value of f.*

2.

$$\langle f, g \rangle = M_{f\overline{g}}$$

The norm given by this inner-product is denoted by $\|f\|$.

is an inner product on AP and with this AP is a pre-Hilbert space.

Exercise: What happens to $M_{f\overline{g}}$ if f and g are actually $2\pi-$periodic continuous functions?

Let us now take $f(t) = \exp(i\lambda t)$ and $g(x) = \exp(i\mu t)$ where λ, μ are real numbers. They are both periodic and hence belong to AP.

Exercise: Check that $\|f\| = 1 = \|g\|$ and that $\langle f, g \rangle = 0$ if $\lambda \neq \mu$ so that the space AP contains the *uncountable family*

$$\{\exp(i\lambda t) \ : \ \lambda \in \mathbb{R}\}.$$

of pairwise orthogonal unit vectors so that the distance between any two of them is $\sqrt{2}$. It follows that the metric space AP cannot be separable and so the Hilbert space completion of AP is also a non-separable Hilbert space.
We shall return to almost periodic functions later if time permits but here are two references.

1. *F. Riesz and B. Nagy, Functional Analysis, Dover NY, Indian Reprint 2007.* See pages 254-260.

2. *C. Corduneanu, Almost periodic functions, Chelsea, New York, 1989.*

Spectral theorem We would like to see the form that the *spectral theorem* takes in a Hilbert space. This is however a long chapter in functional analysis that would take us too far afield. However there is one important case that is relevant to us in the context of *generalized Fourier expansions* namely *regular Sturm-Liouville problems* and Fourier expansions in terms of eigen-functions of a two point boundary value problem for an ordinary differential equation.

We have seen the classical approach in chapter 5 and we now return to this in the Hilbert space setting.

This concerns the spectral theorem for a *compact self-adjoint operator on a Hilbert space*. We begin with the relevant terms and definitions with some examples.

Compact operators on a Hilbert space

Recall that an operator $T : H \longrightarrow H$ is said to be bounded (= continuous) if

$$\|Tx\| \leq C\|x\|, \qquad \text{for all } x \in H.$$

In other words the image $T(U)$ of the closed unit ball U in H is bounded in H. Compact operators are stronger:

Definition: Let H be a Hilbert space. An operator $T : H \longrightarrow H$ is said to be *compact* if the image $T(U)$ of the closed unit ball U in H is *precompact* in H. Note that if H is finite dimensional every operator on H is compact. In general compact operators form a distinguished subclass of operators on a Hilbert space.

We now look at more interesting examples of compact operators on Hilbert spaces. We shall begin with the Hilbert space $L^2[0,1]$

Example 1 (The Volterra operator): $T : L^2[0,1] \longrightarrow L^2[0,1]$ given by

$$Tf(x) = \int_0^x f(t)dt.$$

The operator arises when we try to solve the initial value problem

$$y' = f(x), \quad y(0) = 0.$$

Note that since $f \in L^2[0,1]$ its integral over $[0,1]$ is actually continuous so that Tf is a continuous function. Well, assume $x < y$.

$$|Tf(x) - Tf(y)| \leq \int_x^y |f(t)|dt \leq |x-y|^{1/2}\|f\|$$

We have actually established the Hölder continuity of f.

The Volterra operator is compact: To establish the compactness of the Volterra operator, we assume that $\|f\| \leq 1$ so that

$$|Tf(x) - Tf(y)| \leq |x - y|^{1/2}$$

for all f in the unit ball U of $L^2[0,1]$. Next, $Tf(0) = 0$ so that putting $y = 0$ we get

$$|Tf(x)| \leq 1$$

for all f in the unit ball of $L^2[0,1]$.

We have to show that the set $\{Tf \; : \; f \in U\}$ is precompact which means every sequence has a subsequence converging in L^2 norm. It suffices to show that there is a subsequence converging uniformly since uniform convergence of a sequence of continuous functions on $[0,1]$ implies convergence in L^2 norm.

We now appeal to the classical *Ascoli-Arzela* theorem that we first recall:

The Ascoli-Arzela theorem 1) A family S of continuous functions on $[0,1]$ is said to be equi-continuous if given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in S$,

$$|x - y| < \delta, \quad \text{implies } |f(x) - f(y)| < \epsilon.$$

The point here is that the δ depends only on ϵ and not on the function f. In other words the SAME δ works for ALL the functions in the family S for a given ϵ.

2) A family S of continuous functions on $[0,1]$ is said to be uniformly bounded if there is an $M > 0$ such that for all $f \in S$ and for all $x \in [0,1]$, we have

$$|f(x)| \leq M.$$

Theorem (Ascoli-Arzela): A family S of continuous functions on $[0,1]$ is precompact if and only if it is equi-continuous and uniformly bounded.

Proof of the Ascoli-Arzela theorem is not difficult but we shall not stop to prove it here. A proof is available for example in *Rudin's Principles of mathematical analysis, third edition, 1976.*

It follows at once from the Ascoli-Arzela theorem that for the Volterra operator the family $\{Tf \; : \; f \in U\}$ is precompact establishing that the Volterra operator is compact.