Fourier Analysis and its Applications

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42 Orthonormal bases in Hilbert spaces

Exercises:

24. Use the previous result to prove that the generating function for the sequence $\{T_n(x)\}$ is

$$G(x,t) = \frac{1-tx}{1+t^2-2tx}.$$

25. Use trigonometry to show that $2T_m(x)T_n(x) = T_{m+n} + T_{m-n}$.

26. Show that $T_n(T_m(x)) = T_{mn}(x)$.

27. Prove that

$$\left(\frac{d}{d\cos\theta}\right)^{n-1} \sin^{2n-1}\theta = (-1)^{n-1}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n}\sin n\theta$$

This formula is due to *C. G. J. Jacobi* (1836). See p. 26 ff. of *G. N. Watson, Treatise on the theory of Bessel functions to understand its immense use in special functions.*
Hint: Put $t = \cos\theta$ and show that

$$f(t) = \left(\frac{d}{dt}\right)^{n-1}(1-t^2)^{n-\frac{1}{2}}$$

is a solution of Tchebychev's ODE whereby

$$f(t) = c_n \sin(n\cos^{-1} t).$$

To determine c_n divide both sides by $\sqrt{1-t}$ and let $t \to 1$.

Orthogonal Polynomials in general: The properties established for the Legendre/Laugerre/Hermite/Tcheb Polynomials are rather typical of classical orthogonal systems of polynomials (and some characteristics do carry over to more general orthogonal systems of functions).

1. It is a general fact that if $\{f_n(x)\}$ is a sequence of orthogonal polynomials then the zeros of $f_n(x)$ are real distinct and lie in the interval of orthogonality.

2. The sequence $\{f_n\}$ satisfies a three term recursion formula.

3. The zeros of f_n and f_{n+1} interlace (we have proved this for some systems of orth. polynomials).

4. There is an analogue for the Rodrigues formula for the system of Hermite, Tchebychev, Laguerre and other classical systems of polynomials. Note: The Laugerre polynomials arise when the sequence $1, x, x^2, \ldots$ is subjected to the Gram-Schmidt process with respect to the inner product on $[0,\infty)$ with weight function e^{-x}.

References for orthogonal systems of polynomials For a general discussion the delightful little book *J. Todd, Intro. to the constructive theory of functions, Birkhäuser, 1963* is an excellent place to begin. The properties presented in the last slide are all established in this book. The Tchebycheff polynomials arising in the study of best approximation problem in sup norm is discussed in detail in the book of J. Todd. The book by *Ian. Sneddon, Special functions of mathematical physics and chemistry, Longman Mathematical Texts, Longman, New York, 1980*, contains very instructive list of problems on Legendre Polynomials (see pp. 96-105).

Abstract Fourier analysis in Hilbert spaces.

Suppose H is a Hilbert space and

$$B = \{v_\alpha \ : \ \alpha \in \Lambda\}$$

is an orthonormal basis. At the moment we are not assuming that B is countable. Hilbert spaces with uncountable orthonormal basis are HUGE but these do appear in some applications in Fourier analysis and differential equations such as the theory of almost periodic functions.

See for instance pp 254-255 of Riesz and B. Nagy, Functional Analysis, Dover Reprint 1990

Fourier coefficients and Fourier series However for most of the spaces we shall be dealing with, orthonormal bases will be countable. Suppose $v \in H$, we call the scalar

$$x_\alpha = \langle v, v_\alpha \rangle, \quad \alpha \in \Lambda$$

the α-th Fourier coefficient of v with respect to the orthonormal basis B. We would like to consider the series

$$\sum_{\alpha \in \Lambda} x_\alpha v_\alpha$$

But what is the interpretation of the series? As it stands, the number of terms is not countable !

A lemma on uncountable sums: Suppose $\{a_\alpha \ : \ \alpha \in \Lambda\}$ is a family of non-negative real numbers we define

$$\sum_{\alpha \in \Lambda} a_\alpha \tag{7.20}$$

to be the supremum over the set of all finite sums $a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n}$ where $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ are arbitrary.

Lemma Suppose the sum (7.20) is finite then all but countably many of the summands a_α are zero.

Proof: Suppose not. For each n let

$$E_n = \{a_\alpha \ : \ a_\alpha > 1/n\}$$

The union of all these sets E_n is the set of indices α with $a_\alpha > 0$ and so one of these sets E_N must be uncountable. Now the set of all finite sums

$$a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_n}, \quad a_{\alpha_1}, a_{\alpha}, \ldots, a_{\alpha_n} \in E_N$$

cannot be bounded above and so cannot have a finite supremum which is a contradiction.

We now return to Hilbert spaces.

We now prove the following

Theorem (Existence of Fourier series): Suppose $v \in H$ and x_α ($\alpha \in \Lambda$) are the Fourier coefficients of v with respect to a given orthonormal basis $B = \{v_\alpha \ : \ \alpha \in \Lambda\}$ then

$$\sum_{\alpha \in \Lambda} |x_\alpha|^2 < \infty$$

and all but countably many Fourier coefficients vanish. The series

$$\sum_{\alpha \in \Lambda} x_\alpha v_\alpha$$

converges in H and its sum is v.

Proof: Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be a finite set of indices in Λ and look at

$$v - (x_{\alpha_1} v_{\alpha_1} + x_{\alpha_2} v_{\alpha_2} + \cdots + x_{\alpha_n} v_{\alpha_n})$$

This vector is perpendicular to each of $v_{\alpha_1}, v_{\alpha_2}, \dots, v_{\alpha_n}$ and hence orthogonal to $x_{\alpha_1} v_{\alpha_1} + x_{\alpha_2} v_{\alpha_2} + \cdots + x_{\alpha_n} v_{\alpha_n}$ and so by Pythagorous's theorem, we get

$$\|v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_n} v_{\alpha_n})\|^2 + \|x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_n} v_{\alpha_n}\|^2 = \|v\|^2 \tag{7.21}$$

From this we infer

$$\|x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_n} v_{\alpha_n}\|^2 \leq \|v\|^2$$

The LHS is precisely

$$|x_{\alpha_1}|^2 + |x_{\alpha_2}|^2 + \cdots + |x_{\alpha_n}|^2$$

We conclude that for any finite collection of indices $\alpha_1, \alpha_2, \dots, \alpha_n$ the sum

$$|x_{\alpha_1}|^2 + |x_{\alpha_2}|^2 + \cdots + |x_{\alpha_n}|^2$$

is bounded above by a fixed number $\|v\|^2$ which proves the first part of the theorem and in particular all but countably many Fourier coefficients x_α vanish.

The series now becomes (retaining only terms with non-zero coefficients)

$$x_{\alpha_1} v_{\alpha_1} + x_{\alpha_2} v_{\alpha_2} + x_{\alpha_3} v_{\alpha_3} + \dots \tag{7.22}$$

with say $\alpha_j \neq 0$. To prove that the series (7.9) converges in H, we first observe that

$$|x_{\alpha_1}|^2 + |x_{\alpha_2}|^2 + |x_{\alpha_3}|^2 + \dots$$

converges and hence its partial sums form a Cauchy sequence namely,

$$\lim_{n \to \infty} (|x_{\alpha_n}|^2 + |x_{\alpha_{n+1}}|^2 + \cdots + |x_{\alpha_{n+k}}|^2) = 0, \quad \text{for all } k \in \mathbb{N}.$$

We now show that the partial sums of (7.22) form a Cauchy sequence. Well,

$$\|x_{\alpha_n} v_{\alpha_n} + x_{\alpha_{n+1}} v_{\alpha_{n+1}} + \cdots + x_{\alpha_{n+k}} v_{\alpha_{n+k}}\|^2 = |x_{\alpha_n}|^2 + |x_{\alpha_{n+1}}|^2 + \cdots + |x_{\alpha_{n+k}}|^2$$

and the RHS tends to zero as $n \to \infty$ for all $k \in \mathbb{N}$ and the proof of convergence is done.

We have to now show that the series converges to v. Let $\epsilon > 0$ be arbitrary. There exists scalars $y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_N}$ such that

$$\|v - (y_{\alpha_1} v_{\alpha_1} + y_{\alpha_2} v_{\alpha_2} + \cdots + y_{\alpha_N} v_{\alpha_N})\|^2 < \epsilon^2/2$$

we may assume that N may be assumed as large as we please by taking zero coefficients.

Now let us select N such that

$$|x_{\alpha_{N+1}}|^2 + |x_{\alpha_{N+2}}|^2 + |x_{\alpha_{N+3}}|^2 + \cdots < \epsilon^2/2$$

Now we observe that the vector

$$v - (x_{\alpha_1} v_{\alpha_1} + x_{\alpha_2} v_{\alpha_2} + \cdots + x_{\alpha_N} v_{\alpha_N}) \perp v_{\alpha_j}, \quad \text{for all } j = 1, 2, \ldots, N.$$

whereby

$$v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_N} v_{\alpha_N}) \perp (x_{\alpha_1} - y_{\alpha_1}) v_{\alpha_1} + \cdots + (x_{\alpha_N} - y_{\alpha_N}) v_{\alpha_N}.$$

An application of Pythagorous's theorem gives

$$\begin{aligned}\|v - (y_{\alpha_1} v_{\alpha_1} + \cdots + y_{\alpha_N} v_{\alpha_N})\|^2 = \|v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_N} v_{\alpha_N})\|^2 + \\ |x_{\alpha_1} - y_{\alpha_1}|^2 + \cdots + |x_{\alpha_N} - y_{\alpha_N}|^2\end{aligned}$$

Hence

$$\|v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_N} v_{\alpha_N})\|^2 < \epsilon^2/2$$

Now if $n > N$ then Pythagorous's theorem again,

$$\begin{aligned}\|v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_n} v_{\alpha_n})\|^2 &= \|v - (x_{\alpha_1} v_{\alpha_1} + \cdots + x_{\alpha_N} v_{\alpha_N})\|^2 + (|x_{\alpha_{N+1}}|^2 + \cdots + |x_{\alpha_n}|^2) \\ &< \epsilon^2/2 + \epsilon^2/2 = \epsilon^2\end{aligned}$$

and the proof is complete.

Separable Hilbert space Recall that a separable metric space is a metric space with a countable dense set and a separable Hilbert space is a Hilbert space with a countable dense set D_1 (a Hilbert space is a metric space). Now we select a linearly independent subset $D_2 \subset D_1$ such that

$$\text{Linear span}(D_1) = \text{Linear span}(D_2)$$

Evidently then

$$D_1 \subset \text{Linear span}(D_2)$$

whereby Linear span(D_2) is dense in H. Now subject D_2 to the Gram-Schmidt process and we get an orthonormal set D_3 such that

$$D_1 \subset \text{Linear span}(D_2) = \text{Linear span}(D_3)$$

We infer from this

Theorem: Every separable Hilbert space has a countable orthonormal basis.

Classic examples of separable Hilbert spaces are $L^2[a, b]$ and $L^2(\mathbb{R})$. We have indicated for these specific orthonormal bases namely the Legendre polynomials $a = -1$ and $b = 1$ and the Hermite functions.

How does one produce natural orthonormal bases for a Hilbert space?

It is useful to recall the case of a finite dimensional Hilbert space namely $\mathbb{R}^n$ with the usual dot product. The standard basis do provide an orthonormal basis but it may not be suitable for analysis of a specific problem.

Problem: Identify the quadric $2xy + 2yz + 2zx = 1$ in $\mathbb{R}^3$. The way to go about this is to write the equation of the quadric as

$$\mathbf{x}^T A\mathbf{x}$$

where $\mathbf{x}$ is the column vector with entries x, y, z and A is a real symmetric matrix. Here A is the matrix given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We appeal to the spectral theorem which asserts that there is an orthonormal basis of eigen-vectors of A say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ is an orthogonal matrix such that $P^{-1}AP$ is a diagonal matrix D.

In other words the appropriate orthonormal basis in which the equation of the quadric is to be written is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Now write $\mathbf{x} = P\mathbf{Y}$ and the equation of the quadric transforms into

$$\mathbf{Y}^T D\mathbf{Y} = 1,$$

which is simpler namely $\lambda_1 Y_1^2 + \lambda_2 Y_2^2 + \lambda_3 Y_3^2 = 1$. *We see clearly the role played by the spectral theorem.*