Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

41 Hermite, Laugerre and Techebycheff's polynomials

Exercises:

(3) Show that

$$\exp(-(x-t)^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) e^{-x^2} t^n \tag{7.12}$$

From this we can get the *generating function* for the sequence $\frac{1}{n!}H_n(x)$.

To obtain this expression we begin with the power series

$$\exp(2zx - z^2) = b_0(x) + b_1(x)z + b_2(x)z^2 + \dots$$

It is easy to see that $0!b_0(x) = 1 = H_0(x)$ and $1!b_1(x) = H_1(x)$. We now find a three term recurrence relation for the coefficients $b_j(x)$ and compare it with the three term recurrence relation for the sequence $H_n(x)$. Well, differentiate the equation

$$D\exp(2xz - z^2) = (2x - 2z)\exp(2xz - z^2)$$

n times and set $z = 0$. Here D stands for differentiation with respect to z. We get

$$(n+1)!b_{n+1}(x) = n!(2x)b_n(x) + \binom{n}{1}(-2)(n-1)!b_{n-1}(x).$$

which simplifies to

$$(n+1)b_{n+1}(x) - 2xb_n(x) + 2b_{n-1}(x) = 0.$$

Use induction to complete the argument that $n!b_n = H_n$ for all n.

4. Use Cauchy's formula for the entire function $z \mapsto \exp(2xz - z^2)$ to estimate $H_n(x)/n!$. For simplicity assume x is real positive and take a circle C of radius $R = 2x$ centered around the origin. Then

$$\frac{H_n(x)}{n!} = \frac{1}{2\pi i}\oint_C \frac{\exp(2zx - z^2)\,dz}{z^{n+1}}$$

We shall need some reasonable estimate to exchange a summation and integration using the dominated convergence theorem. An essential step in proving completeness of the Hermite functions.

5. Suppose $|x| \le 1$ show using the three term recurrence relation that

$$|H_n(x)/n!| \le 2^n \tag{7.13}$$

and if $|x| \ge 1$ obtain the estimate

$$|H_n(x)/n!| \le |Cx|^{-n}\exp(x^2/3) \tag{7.14}$$

where C is independent of n and x. Clearly we may assume $x > 0$. For the last part, apply the Cauchy's estimate for the $n-$th derivative of the function $\exp(2xz - z^2)$ taking a circle centered at the origin and of radius R. You get

$$\Big|\frac{H_n(x)}{n!}\Big| \le \frac{1}{\pi R^n}\int_0^{\pi} \exp(2Rx\cos t - R^2\cos 2t)dt \tag{7.15}$$

Crude estimates would suffice and we take $R = x/8$. The integrand in (7.15) can be upper-bounded by

$$\exp(2Rx + R^2) = \exp(\frac{x^2}{4} + \frac{x^2}{64}) < \exp(x^2/3)$$

Thus we immediately get the result

$$\Big|\frac{H_n(x)}{n!}\Big| \leq \frac{1}{\pi R^n}\int_A \exp(2Rx + R^2)dt \leq |Cx|^{-n}\exp(x^2/3), \quad |x| \geq 1.$$

The inequalities (7.13)-(7.14) can be combined into one (weaker) inequality:

$$\Big|\frac{H_n(x)}{n!}\Big| \leq C^n \exp(x^2/3) \tag{7.16}$$

where C is a positive constant. We are now in a position to complete the discussion that we first state as:

Theorem: The linear span of

$$\{H_n(x)\exp(-x^2/2) \ : \ n = 0, 1, 2, \dots\} \tag{7.17}$$

is dense in $L^2(\mathbb{R})$ namely (7.17) is a complete orthogonal system for $L^2(\mathbb{R})$.

To prove this suppose that $f(x) \in L^2(\mathbb{R})$ and $f \perp H_n(x)\exp(-x^2/2)$ for $n = 0, 1, 2, \dots$ so that

$$\int_{\mathbb{R}} f(x)H_n(x)\exp(-x^2/2)dx = 0, \quad n = 0, 1, 2, \dots \tag{7.18}$$

Multiply (7.18) by $t^n/n!$ and sum over n. The exchange of summation and integration needs justification using the DCT. From our discussion on the estimates for $|H_n(x)|/n!$, the partial sums are all dominated by

$$f(x)\exp\Big(\frac{x^2}{3} - \frac{x^2}{2}\Big)\sum_{n=0}^{\infty}(Ct)^n$$

This is in $L^1(\mathbb{R})$ if $|t| < 1/C$ since the other factor rapidly decreasing. Hence using (7.12) we easily get the result:

$$\int_{\mathbb{R}} \exp\Big(-\frac{1}{2}(x-2t)^2\Big)f(x)dx = 0. \tag{7.19}$$

But observe that the LHS of (7.19) is holomorphic as a function of $t \in \mathbb{C}$ and the above holds for all values of t real or complex. Now let G be the Gaussian $\xi \mapsto \exp(-\xi^2/2)$ and (7.19) reads $(G*f)(2t) = 0$. Taking Fourier transform and appealing to the convolution theorem, we get

$$\widehat{f} = 0$$

and so $f = 0$ as desired. The proof is complete.

The Laguerre polynomials and Laguerre functions As two further examples before continuing with the theory, let us look at $L^2(0, \infty)$. This is yet another classical example with applications to quantum mechanics. Here again we shall construct a orthogonal basis of the form $L_n(x)e^{-x/2}$ where $L_n(x)$ are polynomials known as Laguerre polynomials.

As in the case of Hermite functions we begin with the Laguerre differential equation - See *Arthur Beiser's Perspectives in Modern Physics for the background in physics.*

We shall not prove the completeness of the Laguerre functions at this stage.

Laguerre differential equation This is the equation

$$xy'' + (1-x)y' + \lambda y = 0.$$

The equation has a polynomial solution $F_\lambda(x)$ when λ is a non-negative integer. Since the Wronskian of two solutions is singular at the origin, it cannot have TWO linearly independent polynomial solutions. So the polynomial solutions for non-negative integer λ are unique upto scalar multiples.

When $\lambda = n$ is a non-negative integer, the Laguerre functions are defined as $L_n(x)e^{-x/2}$ with $L_n(x)$ a scalar multiple of F_n normalized so that the L^2 norm is one.

The ODE is easily converted into self-adjoint form through multiplication by e^{-x} namely

$$\left(xe^{-x}y'\right)' + \lambda e^{-x}y = 0.$$

From this we immediately infer the orthogonality of the Laguerre functions.

Theorem (Orthogonality of the Laguerre functions): The functions $\{L_n(x)e^{-x/2} \; : \; n = 0, 1, 2, \dots\}$ is an orthogonal system of functions in $L^2(0,\infty)$ namely

$$\int_0^\infty L_n(x)L_m(x)e^{-x}dx = 0, \quad m \neq n.$$

Proof is simple. We have the two equations

$$\left(xe^{-x}L_n'\right)' + ne^{-x}L_n = 0, \quad \left(xe^{-x}y'\right)' + me^{-x}y = 0.$$

Multiply the first by L_m, second by L_n integrate by parts and subtract and the result follows.

We leave the normalization computation to the audience.

Tchebychev's Differential Equation:

12. Discuss the series solutions of the Tchebychev's differential equation:

$$(1-x^2)y'' - xy' + p^2y = 0$$

Show that if p is an integer, of the two linearly independent solutions *exactly* one of them terminates into a polynomial solution which after suitable renormalization is denoted by $T_n(x)$.

13. Rewrite the ODE in self-ajoint form and show that if $k \neq l$

$$\int_{-1}^1 T_k(x)T_l(x)(1-x^2)^{-1/2}dx = 0.$$

In other words the Tchebychev's polynomials form an orthogonal system with respect to the weight function $(1-x^2)^{-1/2}$.

14. Show that $\sin(p\sin^{-1}(x))$ and $\cos(p\cos^{-1}x)$ satisfy the Tchebychev's equation.

15. Show that $T_n(x) = \cos(n\cos^{-1} x)$. This means you need to prove first that the function on the right is a polynomial. Then invoke uniqueness of T_n as a polynomial of degree n satisfying the ODE with appropriate normalization. Assume by induction that

$$\cos nt = \text{Polynomial in } \cos t.$$

So that

$$\cos(n+1)t = \cos nt \cos t - \sin nt \sin t = \text{Polynomial in } \cos t - \sin nt \sin t$$

Now we write $\exp(it) = a$ and we see that

$$-4\sin nt \sin t = (a^n - a^{-n})(a - a^{-1}) = (a^2 + a^{-2} - 2)(\dots) = (2\cos 2t - 2)(\dots)$$

For more problems on Tchebychev's polynomials, the student is referred to pp 177-187 of *L. Sirovich, Introduction to Applied Mathematics, Springer Verlag, 1988.* In the next slide we shall list some from chapter 6 of the book of *L. Sirovich*.

Additional Problems on Tchebychev's polynomials:

20. Recall that $T_n(x) = \cos(n\cos^{-1}(x))$. Use this to determine the three term recursion formula for the sequence $\{T_n(x)\}$.
Ans: $T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$.

21. Compute the integral

$$\int_{-1}^{1} \frac{(T_n(x))^2 dx}{\sqrt{1-x^2}}$$

22. As for the case of Legendre polynomials, show that the Tchebychev's polynomial $T_n(x)$ has n distinct roots in $(-1, 1)$. Determine these roots.

23. Show that

$$T_n(x) = \frac{1}{2}\Big\{(x - i\sqrt{x^2-1})^n + (x + i\sqrt{x^2-1})^n\Big\}$$

Hint: Write cosine in exponential form.

24. Use the previous result to prove that the generating function for the sequence $\{T_n(x)\}$ is

$$G(x,t) = \frac{1 - tx}{1 + t^2 - 2tx}.$$

25. Use trigonometry to show that $2T_m(x)T_n(x) = T_{m+n} + T_{m-n}$.

26. Show that $T_n(T_m(x)) = T_{mn}(x)$.

27. Prove that

$$\left(\frac{d}{d\cos\theta}\right)^{n-1} \sin^{2n-1}\theta = (-1)^{n-1}\frac{1\cdot 3\cdot 5\dots(2n-1)}{n}\sin n\theta$$

This formula is due to *C. G. J. Jacobi* (1836). See p. 26 ff. of *G. N. Watson, Treatise on the theory of Bessel functions to understand its immense use in special functions.*
Hint: Put $t = \cos\theta$ and show that

$$f(t) = \left(\frac{d}{dt}\right)^{n-1}(1-t^2)^{n-\frac{1}{2}}$$

is a solution of Tchebychev's ODE whereby

$$f(t) = c_n \sin(n \cos^{-1} t).$$

To determine c_n divide both sides by $\sqrt{1-t}$ and let $t \to 1$.