Fourier Analysis and its Applications

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40 Completeness of Hermite Functions

Unlike the case of Legendre polynomials, there is no universally accepted convention for normalizing these polynomials. Let us call these polynomials $F_\lambda(x)$ (λ non-negative integer).

Theorem (Orthogonality of Hermite polynomials): The polynomials $F_\lambda(x)$ ($\lambda = 0, 1, 2, \dots$) form an orthogonal family in the following sense:

$$\int_{\mathbb{R}} F_m(x)F_n(x)e^{-x^2}dx = 0, \quad m \neq n.$$

Define

$$h_n(x) = F_n(x)e^{-x^2/2}, \quad n = 0, 1, 2, \dots$$

The family $\{h_n(x) \,:\, n = 0, 1, 2, \dots\}$ is an orthogonal system of functions on $L^2(\mathbb{R})$.

Let us dispose off the proof of the theorem. Multiply the differential equation

$$y'' - 2xy' + 2\lambda y = 0$$

by $\exp(-x^2)$ to make it self-adjoint whereby

$$\frac{d}{dx}\Big(e^{-x^2}y'\Big) + 2\lambda y e^{-x^2} = 0.$$

Now,

$$\begin{aligned} \frac{d}{dx}\Big(e^{-x^2}F_m'\Big) + 2mF_m e^{-x^2} &= 0. \\ \frac{d}{dx}\Big(e^{-x^2}F_n'\Big) + 2nF_n e^{-x^2} &= 0. \end{aligned}$$

Multiply the first by F_n, second by F_m integrate by parts and subtract.

An explicit formula for the Hermite functions We now proceed to obtain an explicit formula for these Hermite functions. Define

$$Q_n(x) = e^{x^2}D^n(e^{-x^2}) \tag{7.7}$$

which is evidently a polynomial of degree n. Now, assume $m < n$. Then,

$$\int_{\mathbb{R}} Q_n(x)Q_m(x)e^{-x^2}dx = \int_{\mathbb{R}} Q_m(x)D^n e^{-x^2}dx$$

Integrate by parts n times transferring all the derivatives onto the factor $Q_m(x)$ and we get

$$\int_{\mathbb{R}} Q_n(x)Q_m(x)e^{-x^2}dx = \int_{\mathbb{R}} D^n Q_m(x)e^{-x^2}dx$$

But $D^nQ_m(x) = 0$ since $m < n$. So we are in the following situation.

We have the vector space V of all polynomials endowed with the innerproduct

$$\langle f(x), g(x)\rangle = \int_{\mathbb{R}} f(x)g(x)e^{-x^2}dx$$

and we have two sets of polynomials

$$Q_0, Q_1, Q_2, \dots, \qquad \text{and} \qquad F_0, F_1, F_2, \dots,$$

such that linear span$\{Q_0, Q_1, Q_2, \dots, Q_n\}$ = linear span$\{F_0, F_1, F_2, \dots, F_n\}$ for EVERY $n = 0, 1, 2, \dots$. By the fundamental orthogonality lemma we infer that there is a sequence of constants c_n such that

$$F_n(x) = c_n Q_n(x), \qquad \text{for every } n. \tag{7.8}$$

Let us briefly recall the statement of the fundamental orthogonality lemma.

Fundamental Orthogonality Lemma restated Suppose V is a vector space endowed with inner product with respect to which $\{v_0, v_1, v_2, \dots\}$ and $\{w_0, w_1, w_2, \dots\}$ are two orthogonal systems of non-zero vectors. Further assume that

$$\operatorname{span}(v_0, v_1, \dots, v_k) = \operatorname{span}(w_0, w_1, \dots, w_k), \qquad \text{for every } k = 0, 1, 2 \dots$$

Then, for certain scalars c_k $(k = 0, 1, 2 \dots)$,

$$v_k = c_k w_k, \qquad \text{for every } k = 0, 1, 2 \dots$$

Normalizing the Hermite polynomials: Now we demand that

$$\int_{-\infty}^{\infty} F_n(x) F_n(x) e^{-x^2} dx = 1.$$

In other words we need to select c_n in such a way that

$$|c_n|^2 \int_{-\infty}^{\infty} Q_n(x) Q_n(x) e^{-x^2} dx = 1.$$

We now compute explicitly the integral

$$\int_{-\infty}^{\infty} Q_n(x) Q_n(x) e^{-x^2} dx$$

from which the value of c_n follows at once.

Well,

$$\begin{aligned} Q_n(x) = e^{x^2} D^n e^{-x^2} &= e^{x^2}(D^{n-1}(-2xe^{-x^2})) \\ &= (-2x)Q_{n-1}(x) - 2(n-1)e^{x^2} D^{n-2} e^{-x^2} \\ &= (-2x)Q_{n-1}(x) + \text{l.o.t} \\ &= (-2x)^2 Q_{n-2}(x) + \text{l.o.t} \\ &= \dots \\ &= (-2x)^n + \text{l.o.t} \end{aligned}$$

Here l.o.t stands for lower order terms. From this we infer that

$$D^n Q_n(x) = (-2)^n n!$$

Now,
$$\int_{-\infty}^{\infty} Q_n(x)Q_n(x)e^{-x^2}dx = \int_{-\infty}^{\infty} Q_n(x)D^n(e^{-x^2})dx$$
Integrating by parts n times we get
$$\begin{aligned}\int_{-\infty}^{\infty} Q_n(x)Q_n(x)e^{-x^2}dx &= (-1)^n\int_{-\infty}^{\infty} D^nQ_n(x)(e^{-x^2})dx \\ &= 2^n n!\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= 2^n n!\sqrt{\pi}\end{aligned}$$

With this we get the normalized Hermite polynomials
$$\frac{1}{2^{n/2}\pi^{1/4}\sqrt{n!}}e^{x^2}D^ne^{-x^2} \tag{7.9}$$

Theorem (Completeness of Hermite functions): Let
$$H_n(x) = (-1)^n e^{x^2}D^ne^{-x^2}, \quad n = 0, 1, 2, \dots \tag{7.10}$$
The family of Hermite functions $\{e^{-x^2/2}H_n(x) \;:\; n = 0, 1, 2, \dots\}$ forms a complete orthonormal basis for $L^2(\mathbb{R})$. We shall not prove the completeness assertion here. These functions are extremely important in Quantum mechanics and the study of the Schrödinger equation. We have indicated in chapter 4 that these functions are eigen-vectors of the Fourier transform operator as an operator on $L^2(\mathbb{R})$. Let us recapitulate this aspect quickly. The differential equation
$$y'' - 2xy' + 2\lambda y = 0$$
goes over to
$$u'' - x^2u + (2\lambda + 1)u = 0$$
where $ye^{-x^2/2} = u$. The u equation has no u' term and so the Wronskian of any two solutions is constant by the Abel-Liouville formula. The other feature is that the u equation is invariant under Fourier transform so that if u is a solution then so is $\widehat{u}$.

Now we know that the Hermite functions $h_n(x) = e^{-x^2/2}H_n(x)$ is a solution of the u equation (with $\lambda = n$) and consequently $\widehat{h_n}$ is also a solution of the same equation. But h_n is evidently in the Schwartz class and so it its Fouirer transform and their Wronskian must be zero (why?).

Hence the functions h_n and $\widehat{h_n}$ must be multiples of each other namely
$$\widehat{h_n} = \lambda_n h_n$$
That is to say each h_n is an eigen vector of the Fourier transform operator from $\mathcal{S}$ onto itself.

What are the eigen-values?

Exercises:

(1) Prove the following *three term recurrence relation* for the sequence of Hermite polynomials $H_n(x)$ given by $H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$.

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{7.11}$$

Write the polynomial $-2xH_n(x)$ as a linear combination $\sum_{j=0}^{n+1} c_j H_j(x)$ and proceed as in the proof of the three term recurrence relation for Legendre polynomials.

(2) Show that the zeros of $H_n(x)$ are all real and distinct. Show that the zeros of $H_n(x)$ and $H_{n+1}(x)$ interlace. Use the self adjoint form

$$\frac{d}{dx}(\exp(-x^2 H_n')) + 2n\exp(-x^2)H_n = 0.$$

Imitate the proof of the corresponding theorem for Legendre polynomials.

(3) Show that

$$\exp(-(x-t)^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) e^{-x^2} t^n \tag{7.12}$$

From this we can get the *generating function* for the sequence $\frac{1}{n!}H_n(x)$.

To obtain this expression we begin with the power series

$$\exp(2zx - z^2) = b_0(x) + b_1(x)z + b_2(x)z^2 + \dots$$

It is easy to see that $0!b_0(x) = 1 = H_0(x)$ and $1!b_1(x) = H_1(x)$. We now find a three term recurrence relation for the coefficients $b_j(x)$ and compare it with the three term recurrence relation for the sequence $H_n(x)$. Well, differentiate the equation

$$D\exp(2xz - z^2) = (2x - 2z)\exp(2xz - z^2)$$

n times and set $z = 0$. Here D stands for differentiation with respect to z. We get

$$(n+1)!b_{n+1}(x) = n!(2x)b_n(x) + \binom{n}{1}(-2)(n-1)!b_{n-1}(x).$$

which simplifies to

$$(n+1)b_{n+1}(x) - 2xb_n(x) + 2b_{n-1}(x) = 0.$$

Use induction to complete the argument that $n!b_n = H_n$ for all n.

4. Use Cauchy's formula for the entire function $z \mapsto \exp(2xz - z^2)$ to estimate $H_n(x)/n!$. For simplicity assume x is real positive and take a circle C of radius $R = 2x$ centered around the origin. Then

$$\frac{H_n(x)}{n!} = \frac{1}{2\pi i}\oint_C \frac{\exp(2zx - z^2)\,dz}{z^{n+1}}$$

We shall need some reasonable estimate to exchange a summation and integration using the dominated convergence theorem. An essential step in proving completeness of the Hermite functions.