Fourier Analysis and its Applications
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04 The ubiquitous Gaussian

Let us look at a simple case where this theorem is applicable. Consider the function

$$f(x) = |x|, \quad |x| \leq \pi$$

extended as a 2π periodic function. Sketch the graph of the function and check that the function is Lipschitz. Since the function is an even function,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0.$$

Exercise: Determine all the Fourier coefficients using formulas (1.4)-(1.5) and deduce that

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{\infty}\frac{4\cos(2k-1)x}{\pi(2k-1)^2}$$

What do you get when $x = 0$?

Partial fraction expansions for trigonometric functions: Begin with the even function $f(x) = \cos(ax)$ on $[-\pi, \pi]$ where $a \notin \mathbb{Z}$. The 2π periodic extension is evidently Lipschitz and we can appeal to the basic convergence theorem. Determine the Fourier coefficients of $f(x)$ and

(i) Show that

$$\operatorname{cosec}(\pi a) = \frac{1}{\pi a} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2 - n^2} \tag{1.19}$$

(ii) Show that

$$\cot(\pi a) = \frac{1}{\pi a} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2 - n^2} \tag{1.19'}$$

(iii) Show that

$$\coth(\pi a) = \frac{1}{\pi a} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2 + n^2} \tag{1.19''}$$

The ubiquitous Gaussian and its Fourier transform: Let us begin by recalling the famous integral:

$$\int_{-\infty}^{\infty}\exp(-x^2)dx = \sqrt{\pi} \tag{1.20}$$

You no doubt have seen the standard evaluation of this integral.

Exercises: An excellent project would be to look up the various proofs of this important identity. Here are some suggestions:

(i) Start with setting $x = \sqrt{n}\tan\ \theta$ in the integral

$$\int_0^{\infty}\Big(1 + \frac{x^2}{n}\Big)^{-n} dx$$

Evaluate the integral in closed form and appeal to *Walli's product formula* for $\sqrt{\pi}$.

(ii) Prove Walli's product formula by looking at the chain of inequalities:

$$\int_0^{\pi/2} \sin^{2n+1}\theta d\theta \leq \int_0^{\pi/2} \sin^{2n}\theta d\theta \leq \int_0^{\pi/2} \sin^{2n-1}\theta d\theta \tag{1.21}$$

A natural question is whether the integral (1.20) can be evaluated via *Cauchy's method of residues* . See the interesting comments on this in *R. Remmert, Theory of functions, Springer-Verlag, 1991* pp. 413-414. An interesting proof of the evaluation via residues was given my *L. Mirsky* , The Mathematical Gazette, Vol. 33 (1949), p. 279.

(iii) Integrate $e^{i\pi z^2}\text{cosec}(\pi z)$ along a parallalogram with vertices $R+\frac{1}{2}+iR$, $R-\frac{1}{2}+iR$, $-R+\frac{1}{2}-iR$ and $-R-\frac{1}{2}-iR$. Use Cauchy's theorem to compute the value of the integral (1.21).

See p. 250 of the book *H. A. Priestly, Introduction to complex analysis, Oxford University Press, second edition, 2005* .

We shall now turn to the computation of the *Fourier transform* of the Gaussian.

Theorem:

$$\int_{-\infty}^{\infty} \exp(-ax^2)\cos\xi x dx = \frac{\sqrt{\pi}}{\sqrt{a}}\exp(-\xi^2/4a) \tag{1.22}$$

To prove this, let us denote the integral on the left hand side as $I(\xi)$. Differentiating with respect to ξ we get

$$I'(\xi) = -\int_{-\infty}^{\infty} x\exp(-ax^2)\sin\xi x dx = \frac{1}{2a}\int_{-\infty}^{\infty}\frac{d}{dx}(\exp(-ax^2))\sin\xi x dx$$

Integrating by parts we get

$$I'(\xi) + \frac{\xi}{2a}I(\xi) = 0.$$

Solving this linear ODE we get

$$I(\xi) = I(0)\exp(-\xi^2/4a).$$

Now evaluate $I(0)$ and complete the proof.

Jacobi Theta Function Identity We shall now give another application of the basic convergence theorem and derive a beautiful identity that is of immense use in *number theory* . Let us consider the function:

$$f(t) = \sum_{n=-\infty}^{\infty}\exp(-(t+2\pi n)^2) \tag{1.23}$$

Exercises: (i) Show that this function is infinitely differentiable.

(ii) We need to interchange limits and integrals. Look up the conditions permitting this in *Rudin's Principles of Math. Analysis* .

So this is a smooth even 2π periodic function of t and we can apply the basic convergence theorem. Let us compute the Fourier coefficients of this function.

$$2\pi a_0 = \int_{-\pi}^{\pi} f(t)dt = \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}\exp(-(t+2\pi n)^2)dt$$

Put $t + 2\pi n = u$ in the integral and we get

$$2\pi a_0 = \sum_{n=-\infty}^{\infty} \int_{\pi(2n-1)}^{\pi(2n+1)} \exp(-u^2) du = \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Exercise: Compute the Fourier coefficients a_n using (1.22). Obviously $b_n = 0$ for all n. Check that the Fourier series for $f(t)$ is given by

$$f(t) = \frac{1}{2\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp(-n^2/4) \cos nt.$$

So we have obtained the identity:

$$\sum_{n=-\infty}^{\infty} \exp(-(t + 2\pi n)^2) = \frac{1}{2\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp(-n^2/4) \cos nt \tag{1.24}$$