Fourier Analysis and its Applications

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39 Hilbert Space Basics

Existence of a cont. funct. whose Fourier series diverges at the origin: The assumption says that $S_N(f,0) = T_N f$ converges as $N \to \infty$ and so the sequence $\{T_N f\}$ is bounded for each $f \in \text{Per}[-\pi,\pi]$. By Banach Steinhaus there must exist $M > 0$ such that

$$|T_N f| \leq M, \quad \text{for all } f \in \text{Per}[-\pi,\pi] \quad \text{with } \|f\| \leq 1.$$

In other words

$$\Big|\int_{-\pi}^{\pi} f(t) D_N(t) dt\Big| \leq M \quad \text{for all } n \in \mathbb{N} \tag{7.3}$$

for all functions $f \in \text{Per}[-\pi,\pi]$ with $-1 \leq f \leq 1$. We restrict to real valued functions. To carry on the discussion further we need the following important information

$$\int_{-\pi}^{\pi} |D_n(t)| dt \sim c \log n \tag{7.4}$$

Let us assume this for the moment and proceed further.

Suppose in (7.3) we take $f(x)$ to be the *signum function* denoted $\sigma(t)$ which takes values ± 1 namely taking value $+1$ on those subintervals where $D_N(t)$ is positive and -1 on those subintervals where $D_N(t)$ is negative. Then (7.3) would read

$$\int_{-\pi}^{\pi} |D_N(t)| dt \leq M, \quad \text{for all } n \in \mathbb{N} \tag{7.5}$$

which plainly contradicts (7.4). *The only objection to this reasoning is that the function that takes only values* ± 1 *is not continuous !* Observe that the sign of $D_N(t)$ alternates in alternate intervals of length π/N so that the signum function we have chosen alternates between -1 and 1 on successive intervals of length π/N and as such lies in $L^1[-\pi,\pi]$. But we can appleal to Luzin's theorem and obtain our signum function σ as a limit of a sequence f_j of continuous functions (vanishing at the endpoints) on $[-\pi,\pi]$ converging to f in L^1 norm.

Thus we have

$$\int_{-\pi}^{\pi} |f_j(t) - \sigma(t)| dt \longrightarrow 0, \quad j \to \infty$$

Hence,

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_N(t)| dt &= \int_{-\pi}^{\pi} D_N(t)\sigma(t) dt \\
&= \int_{-\pi}^{\pi} D_N(t)(\sigma(t) - f_j(t)) dt + \int_{-\pi}^{\pi} D_N(t) f_j(t) dt \\
&\leq \int_{-\pi}^{\pi} |D_N(t)||\sigma(t) - f_j(t)| dt + \Big|\int_{-\pi}^{\pi} D_N(t) f_j(t) dt\Big| \\
&\leq N \int_{-\pi}^{\pi} |\sigma(t) - f_j(t)| dt + M
\end{aligned}$$

Now for N fixed we let $j \to \infty$ and we get

$$\int_{-\pi}^{\pi} |D_N(t)| dt \leq M$$

which is a contradiction since the LHS becomes arbitrarily large.

Now we need to establish the following:

$$L_n = \int_{-\pi}^{\pi} |D_n(t)| dt \sim c \log n \tag{7.4}$$

The constants L_n are called the Lebesgue constants and a great deal of information concerning its asymptotic behaviour is known today. However we shall not concern ourselves with these finer points of classical Fourier analysis.

Behaviour of the Lebesgue constants: Clearly it suffices to look at $[0, \pi]$. We have

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin(t/2)}$$

and this changes sign at points

$$\frac{k\pi}{n + \frac{1}{2}}, \quad k = 0, 1, 2, \ldots$$

which divides the interval $[0, \pi]$ into finitely many subintervals each of length $\pi/(n + \frac{1}{2})$. On each of these intervals we have the estimate

$$\frac{1}{\sin(t/2)} \geq \frac{2}{t}$$

and so we have

$$|D_n(t)| \geq \frac{2}{t}\Big|\sin t\Big(n + \frac{1}{2}\Big)\Big|$$

Now on the smaller sub-intervals of length $\pi/2(n + \frac{1}{2})$

$$I_k = \{t \ : \ \pi\Big(\frac{k + \frac{1}{4}}{n + \frac{1}{2}}\Big) \leq t \leq \pi\Big(\frac{k + \frac{3}{4}}{n + \frac{1}{2}}\Big)\}$$

we have the estimates:

$$\Big|\sin t\Big(n + \frac{1}{2}\Big)\Big| \geq \frac{1}{\sqrt{2}}$$

So that

$$|D_n(t)| \geq \frac{1}{t} \geq \frac{2}{\pi}\frac{2n + 1}{4k + 3} \quad \text{ on } I_k.$$

Integrating over I_k we get the estimate

$$\int_{I_k} |D_n(t)| dt > 2/(4k + 3)$$

Summing over $k = 1, 2, \ldots, n$ we get the estimate

$$\int_0^{\pi} |D_n(t)| dt \geq C(1 + \frac{1}{2} + \cdots + \frac{1}{n}) \geq \tilde{C} \log n.$$

completing the demonstration.

Hilbert space basics:

Recall that a Hilbert space H is a vector space endowed with an inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ such that the distance function

$$d(\mathbf{v}, \mathbf{w}) = (\langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle)^{1/2}$$

is complete. The norm of a vector $\mathbf{v}$ is by definition

$$\|\mathbf{v}\| = (\langle \mathbf{v}, \mathbf{v} \rangle)^{1/2}$$

In chapter 2 we have already enlisted a few properties of the inner product such as the Pythagorous's theorem and the parallelogram identity. We now embark upon a deeper study of Hilbert spaces.

Vectors $\mathbf{v}$ and $\mathbf{w}$ in a Hilbert space H are said to be *perpendicular* or *orthogonal*, denoted by $\mathbf{v} \perp \mathbf{w}$ if

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0$$

The orthocomplement of a vector subspace. Suppose H is a Hilbert space and V is a vector subspace of H we define $V^{\perp}$ (the orthocomplement of V) to be

$$V^{\perp} = \{x \in H \ : \ \langle x, \mathbf{v} \rangle = 0, \quad \text{for all } \mathbf{v} \in V\}$$

Exercise: Check that $V^{\perp}$ is a vector subspace of H and $V \cap V^{\perp} = \{0\}$.

Evidently $\{0\}^{\perp} = H$ and $H^{\perp} = \{0\}$.

Exercise: Show that $W^{\perp}$ is always a closed subspace of H and that $V^{\perp\perp} = \overline{V}$.

Exercise: Show that any set of non-zero orthogonal vectors is linearly independent.

Exercise: Discuss the Gram-Schmidt orthonormalization process from linear algebra.

Basis and Hamel Basis In the context of infinite dimensional Hilbert spaces, the ordinary notion of basis in the sense of linear algebra is rarely useful. Let us once and for all emphasize that basis in the sense of linear algebra will always be called *Hamel Basis*. The word basis (without the adjective "Hamel") means something quite different that we now define.

A *basis* in a Hilbert space is a *linearly independent* subset B of H such that the *closure* of the linear span of B is the whole of H.

Remark: A Hilbert space *cannot have a countable Hamel basis* unless it is finite dimensional. This follows from the Baire Category theorem. Since we shall make no use of this result we shall not prove it here in detail.

Recall that a metric space is separable if it has a countable dense subset. The same applies to a Hilbert space - after all a Hilbert space is a metric space !

Hamel bases is HUGE The cardinality of a Hamel basis is uncountable in any infinite dimensional Hilbert space. This can be seen as follows. Suppose the Hamel basis is countable.

(i) A finite dimensional subspace of a Hilbert space is always closed and has empty interior if the Hilbert space has infinite dimension.

(ii) So if the Hamel basis is countable say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ then we can look at

$$E_n = \text{Linear span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

which is finite dimensional and closed and so has empty interior.

(iii) So if the Hamel basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ generates the whole space,

$$H = \bigcup_{n=1}^{\infty} E_n$$

contradicting the Baire category theorem.

Examples of Bases: (1) The classic example of a Hilbert space is $L^2[a,b]$. If we take $a = -\pi, b = \pi$ then we know that the partial sums of the Fourier series of $f \in L^2[-\pi,\pi]$ converges in L^2-norm to f and since the partial sums of the Fourier series are finite linear combinations of

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots,$$

we infer that the linear span of these is dense in $L^2[-\pi,\pi]$ namely these functions forms a basis for $L^2[-\pi,\pi]$. This is an orthogonal basis. One can of course normalize these and obtain a countable orthonormal basis.

(2) As another example let us take $L^2[-1,1]$. We know that polynomials are dense in $C[-1,1]$ and since convergence in sup-norm implies convergence in L^2-norm we infer that every continuous function is a limit of a sequence of polynomials in converging in L^2-norm. Now Luzin's theorem implies that every $f \in L^2[-1,1]$ is the limit of a sequence of polynomials in L^2-norm.

Thus the list

$$1, x, x^2, \dots,$$

provides a countable basis for $L^2[-1,1]$. Now, the example in (2) is a countable basis but not an orthogonal basis for $L^2[-1,1]$.

(3) If we orthonormalize the set $1, x, x^2, \dots$ using the Gram-Schmidt process then we get a countable orthogonal basis for $L^2[-1,1]$. *What is the resulting system of polynomials?*

We get the sequence of Legendre polynomials !

Exercise: Prove this using the fundamental orthogonality lemma in module 5.

(4) **The Hermite functions:** Recall that the Hermite polynomials are solutions of the Hermite's differential equation:

$$y'' - 2xy' + 2\lambda y = 0$$

Let us seek a power series solution of this differential equation

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{7.6}$$

So that (proceeding as in the case of the Legendre equation)

$$\sum_{n=0}^{\infty} x^n \Big(a_{n+2}(n+2)(n+1) - 2na_n + 2\lambda a_n \Big) = 0.$$

Equating the coefficients to zero we get the recurrence relation:

$$a_{n+2} = \frac{2(\lambda - n)}{(n+1)(n+2)} a_n$$

Setting $n = 0, 1, 2, \dots$ we get the complete solution

$$\begin{aligned} y(x) &= a_0\left\{1 + \frac{2\lambda}{2!}x^2 + \frac{2^2\lambda(\lambda-2)}{4!}x^4 + \dots\right\} \\ &\quad + a_1\left\{x + \frac{2(\lambda-1)}{3!}x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5 + \dots\right\} \end{aligned}$$

Observe that if λ is an even integer the first of the bracketed term terminates yielding a polynomial solution. However if λ is an odd integer, the second of the bracketed term terminates yielding a polynomial solution. The polynomial has degree exactly λ when λ is a non-negative integer.