Fourier Analysis and its Applications

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38 The Banach Steinhaus's theorem

Continuous linear maps on Banach Spaces: Suppose X and Y are Banach spaces then we are interested in studying linear transformations $T : X \longrightarrow Y$ that are continuous. The following are equivalent for a linear map $T : X \longrightarrow Y$

(i) T is continuous.

(ii) T is continuous at the origin.

(iii) There is a constant $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$.

(iv) T is uniformly continuous.

To prove (ii) implies (iii), let $\epsilon = 1$. There is a $\delta > 0$ such that $\|x\| < \delta$ implies $\|Tx\| < 1$. Let $x \neq 0$ so that $y = \delta x/2\|x\|$ has norm less than δ and so $\|Ty\| < 1$ from which we get

$$\|Tx\| < 2\|x\|/\delta.$$

So $2/\delta$ is the M we are looking for.

The Baire Category theorem: This is one of the most important result in general topology on which the entire edifice of functional analysis rests. Let us recall this important result.

Theorem: Suppose X is a complete metric space such that

$$X = \bigcup_{n=1}^{\infty} E_n$$

where each E_n is closed then at least one of the sets E_n must have non-empty interior.

Note that a closed set E has empty interior precisely when the complement $X - E$ is a dense open set. With this simple observation the proof is quite easy.

Proof: We prove this by contradiction. Suppose the result is false. Then each $G_n = X - E_n$ is open dense and

$$\bigcap_{n=1}^{\infty} G_n = \emptyset.$$

We shall arrive at a contradiction by showing that the intersection of all these sets G_n is not only non-empty but in fact dense in X. So let $p \in X$ be arbitrary. Since G_1 is dense, the ball B of radius $\epsilon > 0$ centered at p must have a point $z_1 \in G_1$. Since G_1 is open we can find a $r_1 > 0$ such that the *closed ball* S_1 of radius r_1 centered at z_1 is contained in $G_1 \cap B$. We may assume that $r_1 < 1/2$.

Now since G_2 is dense, the open ball $B_{r_1}(z_1)$ must intersect G_2 at say $z_2 \in B_{r_1} \cap G_2$. We select $r_2 > 0$ such that $r_2 < 1/4$ and the *closed ball* S_2 of radius r_2 centered at z_2 is contained in $B_{r_1} \cap G_2$ so that

$$S_2 \subset B_{r_1} \subset S_1$$

Next G_3 is dense in X and so the open ball B_{r_2} intersects G_3 at a point say $z_3 \in B_{r_2} \cap G_3$ and G_3 being open we infer that there is a $r_3 > 0$ such that $r_3 < 1/8$ and *closed ball* S_3 of radius r_3 centered at z_3 is contained in $B_{r_2} \cap G_3$ so that

$$S_3 \subset B_{r_2} \subset S_2 \subset B_{r_1} \subset S_1$$

Proceeding thus we construct a nested sequence

$$S_1 \supset S_2 \supset S_3 \supset \dots$$

such that the diameters of these sets tend to zero. By Cantor's intersection theorem we infer that there is a point q contained in ALL the sets S_n. But for each j by construction

$$S_j \subset G_j$$

which means that q lies in ALL the sets G_j which is a contradiction since the intersection of ALL the sets G_n is empty. We have completed the proof of the Baire category theorem.

Banach Steinhaus's theorem: We shall only need this result in the special case where we have a sequence of bounded linear maps $T_n : X \longrightarrow \mathbb{C}$. We say that a sequence of continuous linear forms $\{T_n\}$ as above is pointwise bounded if for each $x \in X$ there is a constant M_x such that

$$\sup_{n\in\mathbb{N}} |T_n x| \leq M_x$$

We say that the sequence of continuous linear forms $\{T_n\}$ is uniformly bounded if there is a constant M such that

$$\sup_{n\in\mathbb{N}} |T_n x| \leq M, \quad \text{for all} \quad \|x\| \leq 1.$$

Theorem (Banach Steinhaus): If a sequence of of continuous linear maps $T_n : X \longrightarrow \mathbb{C}$ is pointwise bounded then it is uniformly bounded.

Proof: Consider the family of closed sets

$$E_j = \{x \in X \ : \ |T_n x| \leq j, \ \text{ for all } n \in \mathbb{N}\}$$

It is evident that these sets are closed. Let us show that the union of these sets is the whole of X. Well, let $x \in X$. Then we know that there is a constant $M_x > 0$ such that

$$|T_n x| \leq M_x, \quad \text{for all } \ n \in \mathbb{N}.$$

If $j > M_x$ then we get $x \in E_j$ and the claim is established that $\bigcup E_j = X$.

Now the Baire category theorem gives one of the sets E_j say E_J has an interior point p which means there is an $r > 0$ such that $B_r(p) \subset E_J$. Also for this p there is a $M_p > 0$ such that

$$|T_n p| \leq M_p, \quad \text{for all } \ n \in \mathbb{N}.$$

Now for any $\|y\| \leq 1$ the point $p + \frac{r}{2}y$ lies in $B_r(p) \subset E_J$ whereby

$$|T_n(p) + \frac{r}{2} T_n y| \leq J, \quad \text{for all } \ n \in \mathbb{N}.$$

Using triangle inequality we get

$$|T_n y| \leq \frac{2}{r}\Big(J + |T_n p|\Big) \leq \frac{2}{r}\Big(J + M_p\Big)$$

Proof of the Banach Steinhaus theorem is now complete.

Existence of a cont. funct. whose Fourier series diverges at the origin: We shall use the Banach Steinhaus's theorem to establish that the set of all $2\pi-$periodic continuous functions whose Fourier series at the origin diverges is a dense subset of Per$[-\pi, \pi]$. Recall that the $N-$th partial sum of the Fourier series for f is given by

$$S_N(f, x) = \int_{-\pi}^{\pi} f(t) D_N(x - t) dt$$

where $D_N(\xi)$ is the Dirichlet kernel. We shall show that $S_N(f, 0)$ fails to converge (as $N \to \infty$) for a large collection of functions $f \in$ Per$[-\pi, \pi]$. Write $S_N(f, 0) = T_N f$ for simplicity and we have

$$T_N f = \int_{-\pi}^{\pi} f(t) D_N(t) dt$$

Suppose the Fourier series of EVERY function in Per$[-\pi, \pi]$ *converges at the origin.* We shall arrive at a contradiction.

The assumption says that $S_N(f, 0) = T_N f$ converges as $N \to \infty$ and so the sequence $\{T_N f\}$ is bounded for each $f \in$ Per$[-\pi, \pi]$. By Banach Steinhaus there must exist $M > 0$ such that

$$|T_N f| \leq M, \quad \text{for all } f \in \text{Per}[-\pi, \pi] \quad \text{with } \|f\| \leq 1.$$

In other words

$$\Big| \int_{-\pi}^{\pi} f(t) D_N(t) dt \Big| \leq M \quad \text{for all } n \in \mathbb{N} \tag{7.3}$$

for all functions $f \in$ Per$[-\pi, \pi]$ with $-1 \leq f \leq 1$. We restrict to real valued functions. To carry on the discussion further we need the following important information

$$\int_{-\pi}^{\pi} |D_n(t)| dt \sim c \log n \tag{7.4}$$

Let us assume this for the moment and proceed further.

Suppose in (7.3) we take $f(x)$ to be the *signum function* denoted $\sigma(t)$ which takes values ± 1 namely taking value $+1$ on those subintervals where $D_N(t)$ is positive and -1 on those subintervals where $D_N(t)$ is negative. Then (7.3) would read

$$\int_{-\pi}^{\pi} |D_N(t)| dt \leq M, \quad \text{for all } n \in \mathbb{N} \tag{7.5}$$

which plainly contradicts (7.4). *The only objection to this reasoning is that the function that takes only values* ± 1 *is not continuous !* Observe that the sign of $D_N(t)$ alternates in alternate intervals of length π/N so that the signum function we have chosen alternates between -1 and 1 on successive intervals of length π/N and as such lies in $L^1[-\pi, \pi]$. But we can appleal to Luzin's theorem and obtain our signum function σ as a limit of a sequence f_j of continuous functions (vanishing at the endpoints) on $[-\pi, \pi]$ converging to f in L^1 norm.