Fourier Analysis and its Applications

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We now develop some basic notions on Banach spaces. Recall that a metric space is said to be *complete* if every Cauchy sequence converges. A *normed linear space* is a vector space V endowed with a map $\| \cdot \| : V \longrightarrow \mathbb{R}$ called a norm satisfying the following properties:

(i) $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$.

(ii) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

(iii) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| = 0$, for all $\mathbf{v}, \mathbf{w} \in V$.

(iv) $\|t\mathbf{v}\| = |t|\|\mathbf{v}\|$ for $\mathbf{v} \in V$ and t scalar.

Note that V could be a real or a complex vector space. Given a norm on V we define a metric on V as

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$

If this metric is complete we say V is a *Banach space.*

The Banach space $C[a, b]$:

The most important example of a Banach space is the set of all real or complex valued continuous functions on $[a, b]$. We define

$$\|f\| = \sup\{|f(x)| \ : \ x \in [a, b]\}$$

We call this the sup-norm on $C[a, b]$. Since a closed subset of a complete metric space is complete we infer that a closed vector subspace of a Banach space is again a Banach space. In particular the subspace of $C[-\pi, \pi]$ consisting of all $2\pi-$periodic continuous functions on the real line can be regarded as a Banach subspace of $C[-\pi, \pi]$. Since we shall be working with this space let us give it a name and call it

$$\text{Per}[-\pi, \pi].$$

More examples of Banach Spaces. There are numerous examples of Banach spaces and here are some

(i) The space $L^p[0, 1]$ where $1 \leq p \leq \infty$.

(ii) The space $L^2(\mathbb{R})$ that is important in the theory of Fourier transforms.

(iii) If X is any compact metric space then $C(X)$ the space of all continuous complex valued functions on X is evidently a Banach space.

(iv) The set of all continuous functions on the closed unit disc $\{|z| \leq 1\}$ that are holomorphic in the interior forms a Banach space with respect to the sup norm.

(v) Set of all 2π-periodic continuous functions on the real line which are Hölder continuous with exponent α. This is a Banach space with respect to the norm

$$\|f\| = |f(0)| + \sup_{x \neq y} \left| \frac{f(x) - f(y)}{(x - y)^\alpha} \right|$$

Exercise: Prove the last two statements.

The Bergman space: Here we shall work out one example in detail. Let Ω be a connected domain in the complex plane. We let $A(\Omega)$ to be the set of all holomorphic functions on Ω and

$$A^2(\Omega) = A(\Omega) \cap L^2(\Omega).$$

Theorem The space $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and so is complete. That is, it is a Hilbert space. This space is known as the *Bergman space*.

To prove the theorem let f_n be a sequence in $A^2(\Omega)$ converging to $f \in L^2(\Omega)$. We have to show that f is actually holomorphic on Ω. We first establish that f is continuous. Since continuity is a local property we may work in a closed disc $D_{2r}(p) \subset \Omega$. We need to recall the *mean value property* for holomorphic functions namely, if g is holomorphic

$$g(p) = \frac{1}{2\pi} \int_0^{2\pi} g(p + \rho e^{it}) dt, \quad 0 \leq \rho \leq r.$$

Let us multiply by ρ and integrate with respect to ρ over the interval $[0, r]$:

$$\frac{r^2}{2} g(p) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r g(p + \rho e^{it}) \rho d\rho dt$$

From this we infer the *solid mean value theorem*:

$$g(p) = \frac{1}{\pi r^2} \iint_{|z-p| \leq r} g(z) dx dy$$

Exercise: Derive the solid mean value property of harmonic functions assuming the mean value property over spheres.

Now let $z, w \in D_r(p)$ so that $D_r(z)$ and $D_r(w)$ both sit inside $D_{2r}(p)$

$$g(z) - g(w) = \frac{1}{\pi r^2} \iint_{D_r(w) \Delta D_r(z)} g(x + iy) dx dy.$$

Taking absolute values and using the Cauchy-Schwartz inequality we get

$$|g(z) - g(w)| \leq \|g\|_{L^2} \frac{\sqrt{\text{area}(D_r(w) \Delta D_r(z))}}{\pi r^2}. \tag{7.1}$$

Exercise: Draw pictures and show that

$$\text{area}(D_r(w) \Delta D_r(z)) \leq \pi 2r|z - w| + O(|z - w|^2). \tag{7.2}$$

Hint: Inscribe the largest possible disc D in the lens formed by $D_r(z)$ and $D_r(w)$. Use that the area of $D_r(w) \Delta D_r(z)$ is strictly less than $2(\pi r^2 - \text{area}(D))$.

We infer from (7.1)-(7.2) that for $n = 1, 2, 3, \ldots$

$$|f_n(z) - f_n(w)| \leq C_r \|f_n\| \sqrt{|z - w|}, \quad z, w \in D_p$$

where C_r is a constant depending on r and from this we infer that the family $\{f_n\}_n$ is equi-continuous on each disc D_p whereby a subsequence converges uniformly on D_p. This uniform limit must be f (why?) whereby the continuity of f on Ω is established.

Exercise: Use the solid MVP on discs to show that the sequence $\{f_n\}_n$ is uniformly bounded on D_p. Show using dominated convergence theorem that the integral of f over every triangle in D_p is zero and deduce that the limit function f is holomorphic.