Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

36 Sturm-Liouville Problems. Existence of eigen-values

Existence of eigen-values and eigen-functions: Returning to the Sturm-Liouville problem $y'' + \lambda\rho(x)y = 0$ with Dirichlet BC at 0 and 1, let us consider the solution $y(x, \lambda)$ of the ODE with *initial conditions*

$$y(0) = 0, \quad y'(0) = 1.$$

The solution $y(x, \lambda)$ is continuously differentiable with respect to the parameter λ and we are interested in those values of λ such that

$$y(x, \lambda) = 0, \text{ when } x = 1. \tag{6.10}$$

Now $\rho(x)$ is continuous and non-negative. We now assume that it is strictly positive and that m^2 and M^2 are its infimum and supremum respectively.

Idea is to appeal to the Sturm comparison theorem with $y'' + \lambda m^2 y = 0$ and $y'' + \lambda M^2 y = 0$ respectively. Note that for each λ, the equation

$$y(x, \lambda) = 0 \tag{6.11}$$

has a discrete set of zeros

$$\zeta_1(\lambda), \zeta_2(\lambda), \zeta_3(\lambda), \ldots$$

and the zeros are simple. Well, suppose x_0 is a double zero then we have in addition to (6.11) the equation

$$y'(x_0, \lambda) = 0 \tag{6.12}$$

Now the uniqueness clause in the fundamental existence uniqueness theorem for the IVP with zero intial conditions at x_0 implies that the function $y(x, \lambda)$ must be identically zero which is a contradiction.

Continuity of the zeros: We now show that each of the zeros $\zeta_k(\lambda)$ varies continuously with respect to λ. To do this we employ the implicit function theorem. Let us prove the continuity at an arbitrary value λ_0 and call $\zeta_k(\lambda_0) = \zeta_0$. Then we have,

$$y(\zeta_0, \lambda_0) = 0.$$

We also know that $y'(\zeta_0, \lambda_0) \neq 0$ where the prime indicates derivative of $y(x, \lambda)$ with respect to x. The implicit function theorem now tells us that there are intervals J_1 and J_2 of λ_0 and ζ_0 respectively such that we can solve the equation

$$y(x, \lambda) = 0$$

uniquely for each $\lambda \in J_1$ for a value $\zeta(\lambda) \in J_2$. This unique solution is also continuously differentiable with respect to λ. We are now ready to prove the existence of an infinite sequence of eigen-values. We compare the function $y(x, \lambda)$ with $\sin M\sqrt{\lambda}x$. Between any two zeros of $y(x, \lambda)$, there must be a zero of $\sin M\sqrt{\lambda}x$. For small values of λ the function $\sin M\sqrt{\lambda}x$ has no zeros in $(0, 1]$ Whereby the first zero of $y(x, \lambda)$ must be larger than 1.

Now look at large values of λ and compare with $\sin m\sqrt{\lambda}x$. Between two zeros of the latter there must be a zero of $y(x, \lambda)$ and the latter has zeros in $(0, 1]$ for large λ which implies that the first zero of $y(x, \lambda)$ must be in $(0, 1)$.

In other words $\zeta_1(\lambda) > 1$ for small values of λ and $\zeta_1(\lambda) < 1$ for large values of λ. By continuity of $\zeta_1(\lambda)$ we see that there is a value λ_1 such that $\zeta_1(\lambda_1) = 0$ which means

$$y(1, \lambda_1) = 0.$$

and this λ_1 is evidently the first eigen-value of the Sturm Liouville problem. The argument for the second eigen value is similar and proceeds by looking at $\zeta_2(\lambda)$ for small and large values of λ respectively. Proof of the existence of eigen-values is thereby completed.

Zeros of eigen-functions: It is evident from our construction that the first eigen-function has no zeros on the open fundamental interval $(0,1)$. All other eigen-functions must have at least one zero in $(0,1)$. This follows at once from the orthogonality of eigen-functions. The zeros of the eigen-function are called the *nodes* of the eigen-function. It is not difficult to show that the n-th eigen-function has exactly $n-1$ nodes in the fundamental interval.

These notions also make sense for boundary value problems in higher dimensions. In higher dimensions they assume a more spectacular aspect. See *Rayleigh's theory of sound Vol 1 and 2* for details on this. Proofs of many of these results can be found in *Courant-Hilbert's methods of mathematical physics.*

With these remarks we close this chapter.

12. Prove the mean value theorem for integrals:

    Suppose f,g are continuous on $[a,b]$ and $g>0$ on (a,b) show that there is a $c\in(a,b)$ such that

    $$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

    Hint: First prove that if f is continuous

    $$\int_a^b f(x)dx = f(c)(b-a), \qquad \text{for some } \ c\in(a,b).$$

    Now use the integral of g over $[a,x]$ as a variable of integration.

13. Let $u(x)=\sqrt{kx}J_n(kx)$. Show that u satisfies the ODE

    $$u'' = -\Big(k^2 - \Big(\frac{n^2-\frac{1}{4}}{x^2}\Big)\Big)u$$

The last equation suggests that when x is very large $u(x)$ must behave like the sine function and $J_n(kx)$ *must behave like* $\sin kx/\sqrt{kx}$ and as such must have infinitely many zeros. We shall see that this is indeed so if $k>1$. The last condition can be removed later.

14. Let $v(x)=\sin(x-a)$. Show that

    $$\frac{d}{dx}(-vu'+uv') = uv\Big(k^2-1-\Big(\frac{n^2-\frac{1}{4}}{x^2}\Big)\Big)$$

15. Let a be so large that $k^2-1-(n^2-1/4)/x^2>0$ on $[a,a+\pi]$. Integrate the equation obtained in the previous exercise over $[a,a+\pi]$ and use the MVT for integrals with

    $$g(x)=\sin(x-a)\Big(k^2-1-\Big(\frac{n^2-\frac{1}{4}}{x^2}\Big)\Big)$$

    which is positive on $(a,a+\pi)$. So for some $c\in(a,a+\pi)$ we have

    $$-(u(a+\pi)+u(a)) = u(c)\int_a^{a+\pi} v(x)\Big(k^2-1-\Big(\frac{n^2-\frac{1}{4}}{x^2}\Big)\Big)dx$$

Thus we see that $u(a), u(c)$ and $u(a+\pi)$ cannot all have the same sign. Thus u must have a zero in every interval $(a,a+\pi)$ for all $a>>1$. We have proved,

Theorem (Zeros of Bessel's Functions): For $k > 1$, the function $J_n(kx)$ has infinitely many zeros for each $n \geq 0$.

Explain why the condition $k > 1$ can be replaced by $k = 1$ or even $k > 0$? We have seen an application of this theorem to the theory of wave propagation. Another interesting proof via the integral representation is on pp. 76 - 78 of *D. Jackson, Fourier series and orthogonal polynomials, Dover, New York, 2004*. See also *G. N. Watson, Treatise on the theory of Bessel functions, p. 500 ff* for a discussion of the techniques used by L. Euler and Lord Rayleigh to compute the zeros of $J_p(x)$.

Hill's equation and the functions of Mathews I mention in passing that the case of an elliptical membrane has been considered by *Émile Léonard Mathieu* in 1868 and the resulting ODE known as the Mathieu equation:

$$y'' + (a + b\cos 2x)y = 0.$$

where b is given and a is an eigen-parameter. The equation has led to a long and rich chapter in the theory of analytic ODEs, generalized and studied by *G. W. Hill* in 1886 in his researches on Lunar motion. Unfortunately we are not in a position to say anything about these exciting theory in this elementary course!