Fourier Analysis and its Applications

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35 The Dirichlet Principle

Let us consider the problem of minimizing the "energy"

$$\int_0^1 (y'(t))^2 dt \tag{6.4}$$

subject to the condition

$$\int_0^1 y(t)^2 \rho(t) dt = 1. \tag{6.5}$$

where $y(t)$ ranges over continuous piecewise smooth functions with $y(0) = y(1) = 0$ and $\rho(x)$ is a positive continuous function on $[0, 1]$.

The Dirichlet Principle: The minimization problem (6.4)-(6.5) has a twice continuously differentiable solution which corresponds to the smallest eigen-value of the Sturm-Liouville problem

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(1). \tag{6.6}$$

The principal difficulty is in showing

(i) That the minimizer exists and

(ii) The minimizer is twice continuously differentiable.

These are rather deep waters. Motivated by potential theoretic considerations, Riemann uses these ideas to prove his celebrated theorem in complex analysis known today as *The Riemann Mapping Theorem*. The Riemann mapping theorem asserts that a *proper* simply connected open subset of $\mathbb{C}$ can be mapped onto the unit disc in the plane by a bijective holomorphic function.

The Riemann mapping theorem: Riemann's arguments proving the mapping theorem had a profound influence on the development of analysis in the last part of the nineteenth century and early twentieth century. Riemann reduced the problem to a variational problem that we now describe. For simplicity assume that the simply connected region is a bounded region Ω with a smooth boundary $\partial\Omega$. Riemann reduced the result to *minimizing* the *energy* integral

$$I[\phi] = \int_\Omega |\nabla\phi|^2 dxdy$$

where ϕ varies over all once continuously differentiable functions ϕ with a prescribed boundary value along $\partial\Omega$. The solution of the variational problem is the Dirichlet problem for the Laplacian on Ω. The issue is to prove:

The existence of a minimizer and the twice differentiability of the minimizer

8. Verify (6.6) formally. That is to say, if $y_0(x)$ is a twice continuously differentiable minimizer then $y_0(x)$ is an eigen-function of the Sturm-Liouville problem.

9. Try to give an intuitive (non-rigorous) geometric argument that the eigen-function corresponding to the smallest eigen value has no zeros in $(0, 1)$. In physics books this is referred to as the *Fundamental Mode*.

10. Assuming that y_0 is the fundamental mode, we seek to minimize (6.4) subject to the condition (6.5) as well as

$$\int_0^1 y(t)y_0(t)\rho(t)dt = 0. \tag{6.8}$$

then we get an eigen function with eigen value larger than the first one. Why is this so? Why is it the case that this eigen-function has at least one zero in $(0,1)$? Is it geometrically clear that there is precisely one zero in $(0,1)$?

11. How does one continue to construct the successive eigen functions?

As a preparation for the next round of results we insert here a simple result:

Theorem (Zeros of non-trivial solutions are isolated): Let $\rho(x)$ be continuous on the open interval I and suppose $y(x)$ is a non-trivial solution of the second order differential equation

$$y'' + \rho(x)y = 0.$$

Then, the zeros of $y(x)$ cannot have a limit point in I. Proof: Assume that $p_1, p_2, p_3 \ldots,$ is a sequence of zeros converging to $p \in I$. Then by continuity of the function $y(x)$ we must have $y(p) = 0$. Without loss of generality we may assume that the sequence is monotone.

By Rolle's theorem, between p_j and p_{j+1} there is a zero q_j of the derivative $y'(x)$ and evidently $q_j \longrightarrow p$ as $j \longrightarrow \infty$ and by continuity of $y'(x)$ we conclude that $y'(p) = 0$. So $y(x)$ satisfies the intial value problem

$$y'' + \rho(x)y = 0, \quad y(p) = 0, \quad y'(p) = 0.$$

The zero function is also a solution of the same IVP whereby we conclude that $y(x) \equiv 0$ which is a contradiction. The proof is complete.

Thus it makes sense to talk about successive zeros of a non-trivial solution of the differential equation. We now come to an important result known as the Sturm's comparison theorem. We shall use this result to demonstrate the existence of eigen-values and eigen-functions.

Theorem (Sturm's comparison theorem): Suppose ρ and σ are both continuous positive functions on $[0,1]$ and $\rho(x) > \sigma(x)$ for every $x \in [0,1]$. Let $y(x)$ and $z(x)$ be solutions of the pair of ODEs

$$y'' + \rho(x)y = 0, \quad z'' + \sigma(x)z = 0.$$

Between two successive zeros of $z(x)$ there is at least one zero of $y(x)$.

Proof: Suppose a and b are successive zeros of $z(x)$ in $[0,1]$ and we may assume $z(x) > 0$ on (a,b). Suppose that $y(x)$ does not vanish on (a,b) and is positive there. We now imitate the proof of orthogonality of eigen-functions but work on the interval $[a,b]$.

Well multiply the first equation by z, the second by y and integrate over $[a,b]$ and subtract:

$$\int_a^b (zy'' - yz'')dx + \int_a^b y(x)z(x)(\rho(x) - \sigma(x))dx = 0.$$

Integrating by parts the first term we see that the integrals involving $y'z'$ would cancel out leaving:

$$z(b)y'(b) - z(a)y'(a) - (y(b)z'(b) - y(a)z'(a)) + \int_a^b y(x)z(x)(\rho(x) - \sigma(x))dx = 0.$$

Which in view of $z(a) = z(b) = 0$ simplifies to:

$$-y(b)z'(b) + y(a)z'(a) + \int_a^b y(x)z(x)(\rho(x) - \sigma(x))dx = 0. \tag{6.9}$$

Now note that $z(x)$ is positive on (a, b) and vanishes at a which means $z'(a) > 0$. Likewise $z'(b) < 0$ which means

$$-y(b)z'(b) + y(a)z'(a) > 0.$$

The integral term in (6.9) is positive and we arrive at the contradiction that the sum of two positive terms is zero. The proof of the theorem is complete.
See the discussion in *Lord Rayleigh: Theory of Sound - I, pp. 219-222* for a very different proof.

Existence of eigen-values and eigen-functions: Returning to the Sturm-Liouville problem $y'' + \lambda\rho(x)y = 0$ with Dirichlet BC at 0 and 1, let us consider the solution $y(x, \lambda)$ of the ODE with *initial conditions*

$$y(0) = 0, \quad y'(0) = 1.$$

The solution $y(x, \lambda)$ is continuously differentiable with respect to the parameter λ and we are interested in those values of λ such that

$$y(x, \lambda) = 0, \text{ when } x = 1. \tag{6.10}$$

Now $\rho(x)$ is continuous and non-negative. We now assume that it is strictly positive and that m^2 and M^2 are its infimum and supremum respectively.