Fourier Analysis and its Applications

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34 Variational properties of eigen values

Exercises:

2. Show that the eigen-values of
$$y'' + \lambda \rho y = 0, \quad (\rho(x) > 0)$$
with boundary conditions $y(0) = 0 = y(1)$ are positive real numbers.

3. Determine the eigen-values and eigen-functions of the Sturm-Liouville problem
$$y'' - 2y' + (1 + \lambda)y = 0$$
with boundary condition $y(0) = 0, y(1) = 0$.

4. Determine the eigen-values and eigen functions of the Sturm-Liouville problem
$$x^2 y'' + xy' + \lambda y = 0$$
on $[e, 1/e]$ with the periodic boundary conditions:
$$y(1/e) = y(e), \quad y'(1/e) = y'(e).$$

5. Show that the eigen-values of the boundary value problem
$$y'' + \lambda y = 0, \quad y(0) = 0,\ y(1) + y'(1) = 0$$
are given by $\lambda = k^2$ where k satisfies $\tan k + k = 0$. Graphically show that there are infinitely many roots. *To see how this type of BC appears in physical problem, see the example on p. 117, §67 of Byerly's text.*

6. A rigid body is rotated with uniform and fixed angular speed about an axis that is not specified. How would one choose the axis of rotation so as to maximize the rotational Kinetic Energy? Formulate the problem mathematically.

7. *Variational principles underlying eigen-values and eigen vectors/functions.* Suppose A is a $n \times n$ real symmetric matrix, show that the maximum and the minimum of the quadratic function
$$\langle Ax, x \rangle, \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$
are both attained at eigen-vectors and the maximum and minimum values are the largest and smallest eigen-values of A.

Proof of the Min/Max properties of Eigen-values: Let $Q(\mathbf{v}) = \mathbf{v}^T A \mathbf{v}$. This is a quadratic polynomial in n-variables and as such attains its minimum value at some point say $\mathbf{v}_1$ on the unit sphere in $\mathbb{R}^n$. We now *perturb* the vector $\mathbf{v}_1$ to say
$$\mathbf{w} = (\mathbf{v}_1 + \epsilon \mathbf{h}) / \|\mathbf{v}_1 + \epsilon \mathbf{h}\|$$
and compare the values of the quadratic at the two points namely
$$Q(\mathbf{v}_1) \leq Q(\mathbf{w}), \quad \text{for all } \epsilon \quad \text{small enough.}$$

Thus, we get after cross multiplying by $\|\mathbf{v}_1+\epsilon\mathbf{h}\|^2$, the following inequality valid for all small ϵ positive or negative !

$$Q(\mathbf{v}_1)\|\mathbf{v}_1+\epsilon\mathbf{h}\|^2 \le (\mathbf{v}_1+\epsilon\mathbf{h})^T A(\mathbf{v}_1+\epsilon\mathbf{h}).$$

Expanding and cancelling off $Q(\mathbf{v}_1)$ we get

$$2\epsilon Q(\mathbf{v}_1)\mathbf{h}^T\mathbf{v}_1 \le 2\epsilon\mathbf{h}^T A\mathbf{v}_1 + \epsilon^2(\mathbf{h}^T A\mathbf{h} - Q(\mathbf{v}_1)\|\mathbf{h}\|^2).$$

We now divide by ϵ and let $\epsilon \to 0$. Since ϵ may have either sign we get the pair of inequalities

$$\mathbf{h}^T(Q(\mathbf{v}_1)\mathbf{v}_1 - A\mathbf{v}_1) \le 0, \quad \text{and } \mathbf{h}^T(Q(\mathbf{v}_1)\mathbf{v}_1 - A\mathbf{v}_1) \ge 0.$$

Thus we conclude $\mathbf{h}^T(A\mathbf{v}_1 - Q(\mathbf{v}_1)\mathbf{v}_1) = 0$. Since $\mathbf{h}$ is arbitrary we see that

$$A\mathbf{v}_1 = Q(\mathbf{v}_1)\mathbf{v}_1$$

In other words the minimum is attained at an eigen-vector and the minimum value is the corresponding eigen-value.
Exercise: Examine carefully the computations and explain where have we used the fact that A is a symmetric matrix?
To proceed further, let S be the intersection of the unit sphere in $\mathbb{R}^n$ with the hyperplane

$$\mathbf{v}\cdot\mathbf{v}_1 = 0.$$

This is also a closed bounded set and the minimum of $Q(\mathbf{v})$ on S is attained at say $\mathbf{v}_2$ and the corresponding $Q(\mathbf{v}_2)$ is not less than $Q(\mathbf{v}_1)$ (why?). Now we *perturb* $\mathbf{v}_2$ to

$$\mathbf{w} = (\mathbf{v}_2+\epsilon\mathbf{h})/\|\mathbf{v}_2+\epsilon\mathbf{h}\|$$

where h is chosen such that $\mathbf{h}\cdot\mathbf{v}_1 = 0$. Again $Q(\mathbf{v}_2) \le Q(\mathbf{w})$. Multiplying the inequality through by $\|\mathbf{v}_2+\epsilon\mathbf{h}\|^2$, expanding out and cancelling $Q(\mathbf{v}_2)$ we get as before

$$2\epsilon\mathbf{h}^T(Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2) \le \epsilon^2(\mathbf{h}^T A\mathbf{h} - Q(\mathbf{v}_2)\|\mathbf{h}\|^2).$$

Dividing by ϵ and letting $\epsilon \to 0$ we get the pair of inequalities from which we again deduce

$$\mathbf{h}^T(Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2) = 0.$$

This is now valid for all $\mathbf{h}$ such that $\mathbf{h}\cdot\mathbf{v}_1 = 0$. But this is also holds for $\mathbf{h}$ parallel to $\mathbf{v}_1$
Exercise: Verify the last statement.
Hence $Q(\mathbf{v}_2)\mathbf{v}_2 - A\mathbf{v}_2 = 0$. Thus $Q(\mathbf{v})$ attains its minimum over S at an eigen-vector $\mathbf{v}_2$. The minimum value is the corresponding eigen-value. By construction $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$. Now minimize $Q(\mathbf{v})$ over the intersection of the unit sphere in $\mathbb{R}^n$ with the pair hyperplanes

$$\mathbf{v}\cdot\mathbf{v}_1 = 0, \quad \mathbf{v}\cdot\mathbf{v}_2 = 0$$

and the rest of the proof simply writes itself out. The process terminates after we have n orthogonal eigen-vectors of our matrix A. We have proved

Theorem ([Spectral Theorem): A real symmetric matrix has an orthonormal basis of eigen-vectors.

The analogue of the above result for self-adjoint Diff. Eqns with Dirichlet BC is a serious matter that has lead to a huge corpus of mathematical research -

for these mathematical development against a historical back-drop see the introductory parts of *R. Courant: The Dirichlet's principle, conformal mappings and minimal surfaces, Dover Reprint, 2005. See the free preview of the first three pages of introduction on the Internet!*
Let us consider the problem of minimizing the "energy"

$$\int_0^1 (y'(t))^2 dt \tag{6.4}$$

subject to the condition

$$\int_0^1 y(t)^2 \rho(t) dt = 1. \tag{6.5}$$

where $y(t)$ ranges over continuous piecewise smooth functions with $y(0) = y(1) = 0$ and $\rho(x)$ is a positive continuous function on $[0, 1]$.

The Dirichlet Principle: The minimization problem (6.4)-(6.5) has a twice continuously differentiable solution which corresponds to the smallest eigen-value of the Sturm-Liouville problem

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(1). \tag{6.6}$$

The principal difficulty is in showing

(i) That the minimizer exists and

(ii) The minimizer is twice continuously differentiable.

These are rather deep waters. Motivated by potential theoretic considerations, Riemann uses these ideas to prove his celebrated theorem in complex analysis known today as *The Riemann Mapping Theorem*. The Riemann mapping theorem asserts that a *proper* simply connected open subset of $\mathbb{C}$ can be mapped onto the unit disc in the plane by a bijective holomorphic function.