Fourier Analysis and its Applications

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33 Regular Sturm-Liouville problems

There remains the task of proving the following

Theorem (zeros of Bessel functions): For p real and non-negative, the function $J_p(x)$ has infinitely many zeros in $(0, \infty)$.

The proof will be given in the next chapter after we complete the proof of the Sturm comparison theorem.

6. The Vibrating String and regular Sturm Liouville Problems:

Suppose we have a string of length l and density $\rho(x)$ stretched along the line segment $[0, l]$ and clamped at its two ends. The string is set into vibrations and the governing differential equation my be reduced to

$$y'' + \lambda\rho(x)y = 0 \tag{6.1}$$

where λ is a parameter. Since the string being clamped at its ends we have the boundary conditions (BC)

$$y(0) = y(l) = 0. \tag{6.2}$$

A λ for which a non-trivial solution exists is called the eigen-value of the B. V. Problem and the corresponding *non-trivial rm* solution the eigen-functions. The BC (6.2) is referred to as the Dirichlet boundary conditions. Other physical conditions lead to BC of the form

$$y'(0) = y'(l) = 0. \tag{6.3}$$

known as the Neumann BC. Later we shall see another BC that has some different features. We shall *assume that $\rho(x)$ is continuous and positive.* As a simple example let us consider a uniform string of unit density and unit length. The problem to be solved is

$$y'' + \lambda y = 0, \quad y(0) = 0 = y(1).$$

Solving the ODE ($\lambda = 0$ doesn't give non-trivial solutions),

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

The condition $y(0) = 0$ immediately gives $A = 0$ and the condition $y(1) = 0$ gives

$$B\sin\sqrt{\lambda} = 0.$$

Since the solution in question is *non-trivial* , $B \neq 0$ which means

$$\sqrt{\lambda} = \pm\pi, \pm 2\pi, \ldots$$

Thus the solution to the boundary value problem exists only for a discrete set of values of the parameter and the eigen-values form a discrete set.

Theorem (Orthogonality of Eigen Functions): For the boundary value problem on $[0, l]$

$$y'' + \lambda\rho(x)y = 0$$

with either Dirichlet or Neumann boundary conditions the eigen-functions corresponding to distinct eigen-values are orthogonal on $[0, 1]$ with respect to the weight function $\rho(x)$ namely,

$$\int_0^l u(x)v(x)\rho(x)dx = 0.$$

Proof: Suppose λ and μ are distinct eigen-values with eigen-functions u and v. Argument is similar to the orthogonality of Legendre Polynomials. We have the pair of equations:

$$u'' + \lambda\rho(x)u = 0, \quad u(0) = 0 = u(l)$$
$$v'' + \mu\rho(x)v = 0, \quad v(0) = 0 = v(l).$$

Multiply the first by v, second by u, integrate over $[0, l]$ by parts and subtract. Details left to the student.

Exercise: Discuss the orthogonality of eigen-functions for the Neumann BC.

Theorem (Simplicity of the eigen-values): To each eigen-value of the BVP (6.1)-(6.2) with Dirichlet BC, there is only one eigen-function up to scalar multiples.

Assume that u and v are two *linearly independent* eigen-functions with the same eigen value then the linear span of u and v is the set of all solutions of the homogeneous ODE

$$y'' + \lambda\rho(x)y = 0$$

and since both u and v vanishes at 0, it follows that every solution of the ODE vanishes at 0. This is plainly false since the solution of the initial value problem for this ODE with initial conditions

$$y(0) = 1, y'(0) = 0$$

which exists by *Picard's theorem*, does not vanish at 0. Contradiction.

Completeness of eigen-functions We now state a theorem on the completeness of the set of eigen-functions of the Sturm-Liouville problem

$$y'' + \lambda\rho(x)y = 0, \quad y(0) = 0 = y(l).$$

We assume that the function $\rho(x)$ is positive and continuous on $[0, l]$.
The theorem is often stated under stronger hypothesis for example
R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume - I, p. 293.
However
E. C. Titchmarsh in his *Eigen-function expansions associated with second order differential equations, Volume - I, Oxford, Clarendon Press, 1969*, proves it with substantially weaker hypothesis (see page 12).
One can also discuss mean convergence but we shall not do so here.

Theorem: For the Sturm Liouville problem in question, there is an infinite sequence of eigen-values

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

tending to infinity. The set of eigen-functions $\{\phi_n : n = 1, 2, \dots\}$ is complete in the following sense. Each Lipschitz function $f(x)$ on $[0, l]$ can be expanded as a *Fourier series*

$$f(x) = \sum_{j=1}^{\infty} c_j\phi_j(x)$$

which converges for each $x \in (0, l)$. The coefficients are given by

$$c_j = \Big(\int_0^l f(x)\phi_j(x)\rho(x)dx\Big)\Big(\int_0^l \phi_j^2(x)\rho(x)dx\Big)^{-1}$$

Exercises:

2. Show that the eigen-values of
$$y'' + \lambda \rho y = 0, \quad (\rho(x) > 0)$$
with boundary conditions $y(0) = 0 = y(1)$ are positive real numbers.

3. Determine the eigen-values and eigen-functions of the Sturm-Liouville problem
$$y'' - 2y' + (1 + \lambda)y = 0$$
with boundary condition $y(0) = 0, y(1) = 0$.

4. Determine the eigen-values and eigen functions of the Sturm-Liouville problem
$$x^2 y'' + xy' + \lambda y = 0$$
on $[e, 1/e]$ with the periodic boundary conditions:
$$y(1/e) = y(e), \quad y'(1/e) = y'(e).$$

5. Show that the eigen-values of the boundary value problem
$$y'' + \lambda y = 0, \quad y(0) = 0,\ y(1) + y'(1) = 0$$
are given by $\lambda = k^2$ where k satisfies $\tan k + k = 0$. Graphically show that there are infinitely many roots. *To see how this type of BC appears in physical problem, see the example on p. 117, §67 of Byerly's text.*

6. A rigid body is rotated with uniform and fixed angular speed about an axis that is not specified. How would one choose the axis of rotation so as to maximize the rotational Kinetic Energy? Formulate the problem mathematically.

7. *Variational principles underlying eigen-values and eigen vectors/functions.* Suppose A is a $n \times n$ real symmetric matrix, show that the maximum and the minimum of the quadratic function
$$\langle Ax, x\rangle, \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$
are both attained at eigen-vectors and the maximum and minimum values are the largest and smallest eigen-values of A.