Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

32 Laplace's integrals for Legendre polynomials

Fourier-Legendre Series: This is a result similar in spirit to the Fourier Bessel expansion. Rather than state the theorem we give an example due to *Lord Rayleigh, Theory of sound, Volume - II, p. 273*.

$$e^{itx} = \sum_{n=0}^{\infty}(2n+1)i^n\sqrt{\frac{\pi}{2t}}J_{n+\frac{1}{2}}(t)P_n(x). \tag{5.16}$$

Prove by induction

$$J_{n+\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi}n!}\Big(\frac{t}{2}\Big)^{\frac{1}{2}+n}\int_{-1}^{1}(1-x^2)^n\cos tx dx \tag{5.17}$$

Formally deduce the result of Lord Rayleigh. This expansion appears in connection with scattering of plane waves by a spherical obstacle.

Discussion of Rayleigh's expansion: Let us begin by recalling

$$J_p(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+p+1)}\Big(\frac{t}{2}\Big)^{p+2n}$$

so that when $p = 1/2$ we get easily

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t = \frac{1}{\sqrt{\pi}n!}\Big(\frac{t}{2}\Big)^{\frac{1}{2}+n}\int_{-1}^{1}(1-x^2)^n\cos tx dx, \quad (n=0).$$

and it is easy to verify that the following result is true for $n = 0$.

$$J_{n+\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi}n!}\Big(\frac{t}{2}\Big)^{\frac{1}{2}+n}\int_{-1}^{1}(1-x^2)^n\cos tx dx.$$

$$\begin{aligned}
\frac{d}{dt}\Big(t^{-n-\frac{1}{2}}J_{n+\frac{1}{2}}(t)\Big) &= \frac{1}{\sqrt{\pi}n!}\Big(\frac{1}{2}\Big)^{\frac{1}{2}+n+1}\int_{-1}^{1}-2x(1-x^2)^n\sin tx dx \\
&= \frac{1}{\sqrt{\pi}(n+1)!}\Big(\frac{1}{2}\Big)^{\frac{1}{2}+n+1}\int_{-1}^{1}\Big(\frac{d}{dx}(1-x^2)^{n+1}\Big)(\sin tx)dx \\
&= \frac{-t}{\sqrt{\pi}(n+1)!}\Big(\frac{1}{2}\Big)^{\frac{1}{2}+n+1}\int_{-1}^{1}(1-x^2)^{n+1}(\cos tx)dx
\end{aligned}$$

Now recalling from chapter 1 that $(t^{-p}J_p(t))' = -t^{-p}J_{p+1}(t)$ we get after some routine algebra the stated result (5.17). We are now ready to establish the result of Rayleigh on the *Fourier-Legendre expansion of the plane wave* $\exp(itx)$ namely equation (5.16). We begin with the Ansatz

$$e^{itx} = \sum_{n=0}^{\infty}c_nP_n(x)$$

where the coefficients c_n would obviously depend on t. Taking the inner-product with $P_n(x)$ gives

$$c_n\|P_n(x)\|^2 = \int_{-1}^{1}e^{itx}P_n(x)dx.$$

We must now use the Rodrigues' formula for $P_n(x)$ and perform an integration by parts:

$$c_n\|P_n(x)\|^2 = \frac{(it)^n}{2^nn!}\int_{-1}^{1}e^{itx}(1-x^2)^ndx = \frac{(it)^n}{2^nn!}\int_{-1}^{1}(1-x^2)^n\cos tx dx.$$

Finally use (5.17) and the fact that $\|P_n(x)\|^2 = 2/(2n+1)$ and we get (5.16).

Laplace's integral representations: Recall that for the Bessel functions $J_n(x)$ of integral orders we had an integral representation that we derived in chapter 1 from Schlömilch's formula:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)d\theta$$

There is a similar formula for the Legendre polynomials $P_n(x)$ due to Laplace that we shall now derive.

Theorem: For a non-negative integer n,

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2-1}\cos\theta)^n d\theta \tag{5.18}$$

We begin with a simple observation:

$$Q_n(x) = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2-1}\cos\theta)^n d\theta$$

is a polynomial in x for $n = 0, 1, 2, \ldots$. Well, expand using the binomial theorem and the odd powers of $\sqrt{x^2-1}$ will have coefficients

$$\int_0^\pi \cos^{2j+1}\theta d\theta.$$

Verify that $Q_0(x) = 1 = P_0(x)$ and $Q_1(x) = x = P_1(x)$.

It suffices to show that the sequence $Q_n(x)$ satisfies the *same* three term recursion as $P_n(x)$ namely,

$$(n+1)Q_{n+1} - x(2n+1)Q_n + nQ_{n-1} = 0.$$

Write A for $x + \sqrt{x^2-1}\cos\phi$ and expand the integrand $A^{n+1} = (x + \sqrt{x^2-1}\cos\phi)A^n$:

$$A^{n+1} = xA^n + A^n\sqrt{x^2-1}\,\frac{d}{d\phi}\sin\phi$$

Integrate by parts. We get

$$Q_{n+1} = xQ_n + \frac{n}{\pi}\int_0^\pi A^{n-1}(x^2-1)\sin^2\phi d\phi.$$

Now write

$$\begin{aligned}(x^2-1)\sin^2\phi &= (x^2-1) - (\sqrt{x^2-1}\cos\phi)^2 \\ &= (x^2-1) - (A-x)^2 = 2Ax - A^2 - 1.\end{aligned}$$

Thus we get

$$Q_{n+1} = xQ_n - \frac{n}{\pi}\int_0^\pi (A^{n+1} + A^{n-1} - 2xA^n)d\phi$$

which readily translates to

$$(n+1)Q_{n+1} - x(2n+1)Q_n + nQ_{n-1} = 0.$$

For a different proof of this see Byerly, pp. 165-167. There is a remarkable transformation that leads to

Theorem (Laplace's second integral representation): For a non-negative integer n and real values of with $x \geq 1$,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{(x+\sqrt{x^2-1}\cos\theta)^{n+1}}. \tag{5.19}$$

It is a routine matter to verify that the integral on the right hand side makes perfect sense if x is real and non-zero. It is trivial to see that the formula is correct if $x = 1$ and we shall assume that $x > 1$. Also note that even if x is complex and not purely imaginary the denominator in the integrand is bounded away from zero in absolute value and so the integral makes sense as soon as we prescribe the value of $\sqrt{x^2-1}$. All we need to check is that $x + \sqrt{x^2-1}\cos\theta \neq 0$ and so by continuity with respect to θ the claim follows. We now assume $x > 1$ and use the first formula of Laplace

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x+\sqrt{x^2-1}\cos\theta)^n d\theta. \tag{5.18}$$

Perform the change of variables

$$x+\sqrt{x^2-1}\cos\theta = 1/(x+\sqrt{x^2-1}\cos\phi). \tag{5.20}$$

We need to check that this change of variables is licit. We regard θ as a function of ϕ and see if it is well-defined in the first place. Solving (5.20) for $\cos\theta$ gives

$$\cos\theta = -\Big(\frac{\sqrt{x^2-1}+x\cos\phi}{x+\sqrt{x^2-1}\cos\phi}\Big) \tag{5.21}$$

Squaring the RHS of (5.21) we see that it is less than or equal to one so that RHS of (5.21) does indeed define θ uniquely in $[0, \pi]$. We also see that the RHS of (5.21) takes the value 1 if and only if $\phi = \pi$ and takes the value -1 precisely when $\phi = 0$. Denominator of RHS of (5.21) never vanishes as we have seen so that the RHS is differentiable as a function of ϕ. Next, look at the function of two variables:

$$F(\theta,\phi) = x+\sqrt{x^2-1}\cos\theta - 1/(x+\sqrt{x^2-1}\cos\phi)$$

$F_\theta(\theta,\phi) \neq 0$ when $\theta \in (0,\pi)$ so that the implicit function theorem does confirm that θ is indeed a smooth function of $\phi \in (0,\pi)$ and the derivative $\frac{d\theta}{d\phi}$ is given by

$$\sin\theta\frac{d\theta}{d\phi} = \frac{-\sin\phi}{(x+\sqrt{x^2-1}\cos\phi)^2} \tag{5.22}$$

Thus $\frac{d\theta}{d\phi} \neq 0$ on the interval $(0,\pi)$ confirming that θ is a stricly decreasing function of ϕ and the change of variables is licit. Let us now make this substitution (5.20) in the integral (5.18) and we get

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{(x+\sqrt{x^2-1}\cos\phi)^{n+2}}\frac{\sin\phi}{\sin\theta}.$$

Finally using (5.21) we get that

$$\frac{\sin\phi}{\sin\theta} = x+\sqrt{x^2-1}\cos\phi$$

completing the proof of (5.19). For a different proof based on complex function theory is available in *R. Courant and D. Hilbert, Methods of Math. Phy - I, pp. 503-504.*

The integrals (5.18)-(5.19) were given by *Pierre Simon Marquis de Laplace* in his great work on Celestial Mechanics: *Traité de méchanique célèste.* A five volume set reprinted by Chelsea. We next look at a result due to C. Neumann (1862).

A formula of C. Neumann: Use Laplace's integral representation to prove

$$\lim_{n\to\infty} P_n(\cos(x/n)) = J_0(x).$$

Solution: Substituting into Laplace's formula (5.18)

$$P_n(\cos(x/n)) = (\cos(x/n))^n \frac{1}{\pi}\int_0^\pi (1 + i\tan(x/n)\cos\theta)^n d\theta \tag{5.23}$$

The factor $(\cos(x/n))^n$ tends to one as $n \to \infty$. Let us then focus on the integral. Note that if $a(\lambda) \to \alpha$ then by L'Hospital's rule,

$$(1 + \lambda a(\lambda))^{1/\lambda} \longrightarrow e^\alpha$$

Passing to the limit in (5.23) we get

$$\lim_{n\to\infty} P_n(\cos(x/n)) = \frac{1}{\pi}\int_0^\pi e^{ix\cos\theta} d\theta = J_0(x).$$