Fourier Analysis and its Applications

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# 31 Legendre polynomials. Interlacing of zeros

**More Problems on Legendre Polynomials**

5. Prove that $P_n'(1) = \frac{1}{2}n(n+1)$.

6. Prove that $P_{n+1}' - xP_n' = (n+1)P_n$

7. Prove that $(x^2-1)P_n' - nxP_n + nP_{n-1} = 0$.

8. Prove that $P_{n+1}' - P_{n-1}' - (2n+1)P_n = 0$.

9. Prove that $xP_n' - P_{n-1}' = nP_n$.

10. Suppose $x^n = \sum_{j=0}^n c_j P_j(x)$ show that $c_n = 2^n(n!)^2/(2n)!$.

11. Use the method of series solutions to find the power series expansion of $(1+x)^a$ where $a$ is any real number. Hint: Find an ODE satisfied by the function.

**Finding $P'(1)$** Start with Rodrigues formula and we get

$$\begin{aligned} P'(1) &= \frac{1}{2^n n!} D^{n+1}(x-1)^n(x+1)^n\Big|_{x=1} \\ &= \frac{1}{2^n n!}\binom{n+1}{1} D^n(x-1)^n D(x+1)^n\Big|_{x=1} \\ &= \frac{1}{2^n n!}\binom{n+1}{1}\left(n!n(x+1)^{n-1}\right)\Big|_{x=1} \\ &= \frac{1}{2}n(n+1). \end{aligned}$$

**Interlacing of zeros of Legendre functions** We now prove another important result concerning the location of zeros of solutions of the Legendre differential equations. Here we do not assume that the parameter in the differential equation is an integer and the solutions in question need not be polynomials.

**Theorem:** Suppose $0 \le p < q$ and $y(x), z(x)$ are respective (non-trivial) solutions of

$$((1-x^2)y'(x))' + p(p+1)y(x) = 0, \quad ((1-x^2)z'(x))' + q(q+1)z(x) = 0.$$

Then between two successive zeros of $y(x)$ there is at least one zero of $z(x)$.

Let $a, b$ be successive zeros of $y(x)$ where, $-1 < a < b < 1$. We shall imitate the proof of orthogonality but on the interval $[a,b]$. Mult. first equation by $z(x)$, the second by $y(x)$, integrate over $[a,b]$ and subtract. We get

$$\Big(-(1-x^2)y(x)z'(x) + (1-x^2)z(x)y'(x)\Big)\Big|_a^b + (p(p+1) - q(q+1))\int_a^b y(x)z(x)dx = 0.$$

Some of the terms displayed drop out leaving us with

$$\Big\{-(1-a^2)z(a)y'(a)\Big\} + \Big\{(1-b^2)z(b)y'(b)\Big\} + \Big\{(p(p+1)-q(q+1))\int_a^b y(x)z(x)dx\Big\} = 0.$$

We may assume without loss of generality that $y(x) > 0$ on $(a, b)$. We prove the theorem by contradiction assuming that the result is false and that $z(x) > 0$ on $(a, b)$. Let us now look at the sign of each of the three terms (enclosed in braces) in the LHS of the last expression. Evidently

$$\Big\{(p(p+1) - q(q+1)) \int_a^b y(x)z(x)dx\Big\} < 0.$$

Since $y(x)$ is increasing at $x = a$ and decreasing at $x = b$ we get that

$$\Big\{ -(1-a^2)z(a)y'(a)\Big\} \le 0, \quad \text{and} \quad \Big\{(1-b^2)z(b)y'(b)\Big\} \le 0.$$

This is a contradiction since RHS is zero.

Exercise: Can $y(x)$ and $z(x)$ have a common zero? Try to prove that if $0 \le p < q$ then between two successive zeros of $J_p(x)$ there is a zero of $J_q(x)$.

**Generating function for the Legendre Polynomials:** Given a sequence $\{a_n\}$ of real or complex numbers, their generating function is by definition the power series

$$\sum_{n=0}^{\infty} a_n t^n$$

**Theorem:**

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1 - 2xt + t^2}}$$

For connections to potential theory see for instance *A. S. Ramsey, Newtonian Attraction, Camb. Univ. Press, p. 131 ff., or pp 121 - 134 of the more comprehensive and classic treatise of O. D. Kellog, Foundations of potential theory, Dover, New York, 1953.* There are several proofs of this important theorem and we select the one from Courant and Hilbert's monumental treatise *Methods of mathematical physics - I.* Only a sketch of the proof is provided and the student can work out the details. According to exercise 11 in the previous slides, the function $V(x,t) = (1 - 2xt + t^2)^{-1/2}$ can be expanded in an absolutely convergent series

$$V(x,t) = \sum_{n=0}^{\infty} R_n(x) t^n$$

where each $R_n(x)$ is a polynomial of degree exactly $n$ (why?) and the series is valid for $x \in [-1, 1]$ and for a certain interval $|t| < \rho$. Indeed $\rho = 0.4$ would suffice (why?). In particular

$$\frac{1}{\sqrt{1 - 2xu + u^2}\sqrt{1 - 2xv + v^2}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R_j(x)R_k(x)u^j v^k$$

Integrating both sides with respect to $x$ over the range $[-1, 1]$ we get

$$\frac{1}{\sqrt{uv}} \ln\Big(\frac{1+\sqrt{uv}}{1-\sqrt{uv}}\Big) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} u^j v^k \int_{-1}^{1} R_j(x)R_k(x)dx.$$

But the left hand side is (using the logarithmic series we have encountered earlier) given by

$$\frac{1}{\sqrt{uv}} \ln\left(\frac{1+\sqrt{uv}}{1-\sqrt{uv}}\right) = \sum_{n=0}^{\infty} \frac{2u^n v^n}{2n+1}$$

Comparing the coefficient of $u^j v^k$ in the last two expressions we get

$$\int_{-1}^{1} R_j(x) R_k(x) dx = 0, \qquad \text{if } j \neq k.$$

Whereas if $j = k$ then

$$\int_{-1}^{1} (R_j(x))^2 dx = 2/(2j+1).$$

The fundamental orthogonality lemma now implies that $R_j(x) = P_j(x)$ for every $j = 0, 1, 2, \ldots$.

**Remark:** The function $V(x,t)$ is the potential due to a point mass $P$ placed at unit distance from the origin at a point $X$ at distance $t$ from the origin, where $x$ is the cosine of the angle between $OX$ and $OP$.

**Fourier-Legendre Series:** This is a result similar in spirit to the Fourier Bessel expansion. Rather than state the theorem we give an example due to *Lord Rayleigh, Theory of sound, Volume - II, p. 273*.

$$e^{itx} = \sum_{n=0}^{\infty} (2n+1) i^n \sqrt{\frac{\pi}{2t}} J_{n+\frac{1}{2}}(t) P_n(x). \tag{5.16}$$

Prove by induction

$$J_{n+\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi} n!} \left(\frac{t}{2}\right)^{\frac{1}{2}+n} \int_{-1}^{1} (1-x^2)^n \cos tx dx \tag{5.17}$$

Formally deduce the result of Lord Rayleigh. This expansion appears in connection with scattering of plane waves by a spherical obstacle.