Fourier Analysis and its Applications

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30 Properties of Legendre polynomials

Exercises:

1. Compute $\int_{-1}^{1}(P_n(x))^2 dx$

2. Show that $\int_{-1}^{1}(1-x^2)(P_n'(x))^2 dx = 2n(n+1)/(2n+1)$. Hint: Multiply the Diff. Eqn by P_n and integrate by parts.

3. Use Rodrigues formula to prove that the Legendre polynomial of degree n has precisely n distinct roots in the open interval $(-1,1)$. Use Rolle's theorem. Note: The roots were used by *Gauss in 1814 in his famous quadrature formula. See the discussion on pp. 56-69 of S. Chandasekhar, Radiative transfer, Dover Publications, Inc., New York, 1960.*

4. Show that the Legendre polynomials satisfy the three term recursion formula

$$(n+1)P_{n+1} - x(2n+1)P_n + nP_{n-1} = 0.$$

Let us take a few minutes and discuss in detail the exercise in the previous slide. Recall Rolle's theorem which says (in particular) that if f is a polynomial vanishing at a and b then the derivative f' must vanish at least once in (a,b).

Now let us assume that the polynomial f has a double root at a and b. Then we have that

$$f(a) = f'(a) = 0, \quad f(b) = f'(b) = 0.$$

Rolle's theorem as such gives a $c \in (a,b)$ such that $f'(c) = 0$. But now we have that f' vanishes at three points namely, a, c and b. Thus applying Rolle's theorem to f' we conclude that f'' must vanish at least once in (a,c) and on (c,b).

In particular taking $f(x) = (1-x^2)^2$, which has double roots at ± 1, we infer that its second derivative must vanish at two distinct points in $(-1,1)$. In other words $P_2(x)$ has two distinct and *exactly two distinct roots* in $(-1,1)$. Now suppose f is a polynomial with a *triple root* at a and b then show that the third derivative f''' must have at least three distinct roots in $(-1,1)$ and in particular the Rodrigues formula says that $P_3(x)$ has three distinct roots in $(-1,1)$.

Generally for each $n = 0, 1, 2, \ldots$, the polynomial

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

has $n-$distinct roots in $(-1,1)$. The Rodrigues formula immediately gives us the following expression for the leading coefficient of $P_n(x)$

$$\mathrm{Lc}(P_n(x)) = \frac{(2n)!}{2^n (n!)^2} \tag{5.13}$$

Well,

$$\begin{aligned}
\mathrm{Lc}(P_n(x)) &= \frac{1}{2^n n!}\mathrm{Lc}\Big(\frac{d^n}{dx^n}(x^2-1)^n\Big) \\
&= \frac{1}{2^n n!}\mathrm{Lc}\Big(\frac{d^n}{dx^n}(x^{2n})\Big) \\
&= \frac{(2n)(2n-1)\ldots(n+1)}{2^n n!} \\
&= \frac{(2n)!}{2^n (n!)^2}
\end{aligned}$$

Let us now derive the three term recurrence relation (exercise 4):

$$(n+1)P_{n+1} - x(2n+1)P_n + nP_{n-1} = 0. \tag{5.14}$$

Since $\{P_0(x), \dots, P_N(x)\}$ is a basis for the vector space of all polynomials of degree at most N, we see that

$$(2n+1)xP_n(x) = a_0P_0(x) + \cdots + a_{n+1}P_{n+1}(x)$$

which is the Fourier expansion of $xP_n(x)$. Now taking the inner-product of both sides with $P_j(x)$ in $L^2[-1,1]$ we get

$$(2n+1)\langle xP_n(x), P_j(x)\rangle = a_j\|P_j(x)\|^2$$

But this can be rewritten as

$$(2n+1)\langle P_n(x), xP_j(x)\rangle = a_j\|P_j(x)\|^2 \tag{5.15}$$

Suppose $0 \le j \le n-2$ then $xP_j(x)$ is a polynomial of degree at most $n-1$ so that it is a linear combination of

$$P_0(x), P_1(x), \dots, P_{n-1}(x)$$

whereby we see that

$$xP_j(x) \perp P_n(x), \quad j = 0, 1, \dots, n-2.$$

So (5.15) implies

$$a_0 = a_1 = \cdots = a_{n-2} = 0$$

and we get

$$(2n+1)xP_n(x) = a_{n-1}P_{n-1}(x) + a_nP_n(x) + a_{n+1}P_{n+1}(x)$$

Now since $P_n(x)$ and $xP_n(x)$ have opposite pairity whereas $xP_n(x), P_{n-1}(x), P_{n+1}(x)$ have the same pairity we conclude that $a_n = 0$ as well. Hence

$$(2n+1)xP_n(x) = a_{n-1}P_{n-1}(x) + a_{n+1}P_{n+1}(x)$$

Putting $x = 1$ we get at once

$$2n+1 = a_{n-1} + a_{n+1}.$$

To obtain one more relation between a_{n-1} and a_{n+1},

$$(2n+1)\mathrm{Lc}(P_n(x)) = a_{n-1}\mathrm{Lc}(P_{n-1}(x)) + a_{n+1}\mathrm{Lc}(P_{n+1}(x)).$$

The details can be left for you to complete.

Normalization constants Now that we have an orthogonal system of vectors we must see what is the norm of each of the vectors namely the normalizing constants for the Legendre polynomials. To this end let us compute the inner product $\langle P_n(x), P_n(x)\rangle$. Well,

$$4^n(n!)^2\int_{-1}^{1}(P_n(x))^2dx = \int_{-1}^{1}D^n(x^2-1)^nD^n(x^2-1)^ndx$$

We must now repeatedly integrate by parts and transfer all the derivatives from the second factor onto the first as we did earlier to establish the orthogonality of the sequence of polynomials $D^m(x^2-1)^m$ for $m = 0, 1, 2, \ldots$. At the first iterate we get

$$4^n(n!)^2 \int_{-1}^{1} (P_n(x))^2 dx = -\int_{-1}^{1} D^{n+1}(x^2-1)^n D^{n-1}(x^2-1)^n dx + B_1$$

where B_1 is the collection of boundary terms that appear. Let us examine these boundary terms in detail. We have

$$B_1 = D^n(x^2-1)^n D^{n-1}(x^2-1)^n \Big|_{-1}^{1}$$

But notice that since $f(x) = (x^2-1)^n$ has ± 1 as roots of multiplicity n, all the derivatives of f upto and including order $n-1$ must vanish at ± 1 implying that the boundary terms B_1 must vanish. Now one more integration by parts gives:

$$4^n(n!)^2 \int_{-1}^{1} (P_n(x))^2 dx = \int_{-1}^{1} D^{n+2}(x^2-1)^n D^{n-2}(x^2-1)^n dx + B_2$$

where the boundary terms B_2 are

$$B_2 = -D^{n+1}(x^2-1)^n D^{n-2}(x^2-1)^n \Big|_{-1}^{1}$$

which again must vanish and so on. After integrating by parts n times we get

$$4^n(n!)^2 \int_{-1}^{1} (P_n(x))^2 dx = (-1)^n \int_{-1}^{1} D^{2n}(x^2-1)^n (x^2-1)^n dx$$

Observe that $D^{2n}(x^2-1)^n = D^{2n}x^{2n} = (2n)!$ whereby

$$\begin{aligned} 4^n(n!)^2 \int_{-1}^{1} (P_n(x))^2 dx &= (2n)! \int_{-1}^{1} (1-x^2)^n dx \\ &= 2(2n)! \int_{0}^{1} (1-x^2)^n dx \\ &= (2n)! \int_{0}^{1} (1-u)^n u^{-1/2} du \end{aligned}$$

Recall now the beta function $B(p,q)$ $(p, q > 0)$ defined by

$$B(p,q) = \int_{0}^{1} (1-u)^{p-1} u^{q-1} du.$$

So the expression obtained in the last slide can be expressed as

$$4^n(n!)^2 \int_{-1}^{1} (P_n(x))^2 dx = (2n)! B(n+1, 1/2).$$

Invoking the famous beta-gamma relation of Euler,

$$B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q),$$

we get

$$4^n (n!)^2 \int_{-1}^{1} (P_n(x))^2 dx = n!(2n)!\sqrt{\pi}/\Gamma(n+\frac{3}{2}).$$

We have used the fact that $\Gamma(1/2) = \sqrt{\pi}$. To simplify the expression further, we use the relation $\Gamma(x+1) = x\Gamma(x)$ and we get

$$\|P_n\|^2 = \frac{(2n)!}{4^n n!(n+\frac{1}{2})(n-\frac{1}{2})\dots\frac{3}{2}\frac{1}{2}} = \frac{2}{2n+1}$$

For $n = 0$ we get $\|P_0\|^2 = 2$ and for $n = 1$ we get $\|P_1\|^2 = 2/3$ which checks out via direct calculation since $P_0(x) = 1$ and $P_1(x) = x$.

More Problems on Legendre Polynomials

5. Prove that $P_n'(1) = \frac{1}{2}n(n+1)$.
6. Prove that $P_{n+1}' - xP_n' = (n+1)P_n$
7. Prove that $(x^2-1)P_n' - nxP_n + nP_{n-1} = 0$.
8. Prove that $P_{n+1}' - P_{n-1}' - (2n+1)P_n = 0$.
9. Prove that $xP_n' - P_{n-1}' = nP_n$.
10. Suppose $x^n = \sum_{j=0}^{n} c_j P_j(x)$ show that $c_n = 2^n(n!)^2/(2n)!$.
11. Use the method of series solutions to find the power series expansion of $(1+x)^a$ where a is any real number. Hint: Find an ODE satisfied by the function.