Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

03 The Riemann Lebesgue lemma

Behaviour of $D_n(t)$ and inadequacy of mere continuity!

$$\int_{-\pi}^{\pi} |D_n(t)| dt$$

decays as $n \to \infty$ but here **our luck has forsaken us !** In fact the truth is:

$$\int_{-\pi}^{\pi} |D_n(t)| dt \sim c \log n, \qquad \text{as} \;\; n \to \infty$$

for some positive constant c. And we see that there is no way to salvage the argument. Indeed as we know, pointwise convergence **fails** for functions that are merely continuous.

Expanding the numerator in the Dirichlet kernel we get a sum of two terms:

$$D_n(\theta) = \frac{1}{2\pi}(\cos n\theta + \sin n\theta \cot(\theta/2))$$

Likewise the integral (1.14) splits into a sum of two integrals:

$$S_n(f,x) - f(x) = (2\pi)^{-1}\Big\{ \int_{-\pi}^{\pi} \cos nt \; \Delta(t,x) dt + \int_{-\pi}^{\pi} \sin nt \; (\cot \frac{t}{2}) \Delta(t,x) dt \Big\} \tag{1.15}$$

where $\Delta(x,t) = f(x-t) - f(x)$.

The Riemann-Lebesgue lemma The first of the two integrals on the Right hand side of (1.15) is easy to deal with in view of the following result known as the *Riemann-Lebesgue lemma.*

Theorem: If $g \in L^1[a,b]$ then we have

$$\lim_{n\to\infty} \int_a^b \cos nt \; g(t) dt = 0, \quad \lim_{n\to\infty} \int_a^b \sin nt \; g(t) dt = 0 \tag{1.16}$$

We shall prove the Riemann-Lebesgue lemma later and we return to equation (1.15) of the previous slide and observe that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \cos nt \; \Delta(t,x) dt = 0$$

So to complete the proof that $S_n(f,x) \longrightarrow f(x)$ as $n \to \infty$ all we need is to secure that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin nt \; \cot(\frac{t}{2}) \Delta(t,x) dt = 0 \tag{1.17}$$

Exercise: The function defined on $[-\pi,\pi]$ via $F(t) = \cot(\frac{t}{2}) - \frac{2}{t}$ if $t \neq 0$ and $F(0) = 0$ is continuous and so by Riemann Lebesgue lemma

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} F(t) \sin nt dt = 0.$$

Adding and subtracting $2/t$ in the integral (1.17) and using the above, all we need to do now is to prove the following:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} t^{-1} \sin nt \; (f(x-t) - f(x)) dt = 0 \tag{1.18}$$

In order to establish (1.18) we need some additional hypothesis on the function $f(x)$. Just assuming continuity would **not** suffice and it is here that the notion of *Hölder continuity* comes into play.

Definition: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be of Hölder class $\alpha > 0$ if there is a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|^\alpha, \quad x, y \in \mathbb{R}.$$

Exercise: Show that if f is of Hölder class $\alpha > 1$ then f is constant. Functions of Hölder class $\alpha = 1$ are called *Lipschitz functions* .

We are now ready to state and complete the proof of the basic convergence theorem.

Theorem: Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is 2π periodic and Hölder continuous to order $\alpha > 0$ then the Fourier series of $f(x)$ converges pointwise everywhere to $f(x)$.

Well, all we need is to show that:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} t^{-1}(f(x-t) - f(x)) \sin(nt)\, dt = 0$$

We split the integral as (again with $\Delta(t, x) = f(x - t) - f(x)$):

$$\int_{-\pi}^{\pi} t^{-1}\Delta(t, x)\ \sin(nt)\ dt = \int_{|t|<\delta} t^{-1}\Delta(t, x)\ \sin(nt)\ dt + \int_{\delta\leq|t|\leq\pi} t^{-1}\Delta(t, x)\ \sin(nt)\ dt.$$

We use the Hölder continuity in the first piece.

$$\begin{aligned}
\Big|\int_{-\pi}^{\pi} t^{-1}\Delta(t, x)\ \sin(nt)\ dt\Big| &= \Big|\int_{|t|<\delta} t^{-1}\Delta(t, x)\ \sin(nt)\ dt + \int_{\delta\leq|t|\leq\pi} t^{-1}\Delta(t, x)\ \sin(nt)\ dt\Big| \\
&\leq L\int_{|t|<\delta} \frac{dt}{|t|^{1-\alpha}} + \Big|\int_{\delta\leq|t|\leq\pi} t^{-1}\Delta(t, x) \sin nt\ dt\Big| \\
&\leq \frac{2L}{\alpha}\delta^\alpha + \Big|\int_{\delta\leq|t|\leq\pi} t^{-1}\Delta(t, x) \sin nt\ dt\Big|
\end{aligned}$$

Now let $\epsilon > 0$ be arbitrary. Select the above $\delta > 0$ such that $2L\alpha^{-1}\delta^\alpha < \epsilon/2$ and we have

$$\Big|\int_{-\pi}^{\pi} t^{-1}\Delta(t, x)\ \sin(nt)\ dt\Big| < \frac{\epsilon}{2} + \Big|\int_{\delta\leq|t|\leq\pi} t^{-1}\Delta(t, x) \sin nt\ dt\Big|$$

By Riemann Lebesgue lemma, there is an n_0 such that second piece is less than $\epsilon/2$ for all $n > n_0$. The proof is thereby complete.

Theorems of K. Weierstrass and N. N. Luzin We recall two important approximation theorems from analysis. The first is classic namely, the *Weierstrass's approximation theorem* :

Theorem 6: Continuous functions on a closed bounded interval $[a, b]$ can be uniformly approximated by polynomials. In other words given a continuous function $f \in C[a, b]$ and $\epsilon > 0$, there exists a polynomial P such that

$$\sup_{a\leq x\leq b} |f(x) - P(x)| < \epsilon$$

We now state the *theorem of Luzin* :

Theorem 7: If $f \in L^1[a,b]$ and $\epsilon > 0$ then there exists a continous function $g \in C[a,b]$ such that

$$\|f - g\| < \epsilon.$$

Proof of the Riemann Lebesgue lemma: The proof proceeds in four easy steps delineated below. Let $g \in L^1[a,b]$. Let $\epsilon > 0$ be arbitrary.

(i) Verify the Riemann Lebesgue lemma for the case $g(t) = t^k$ and hence for all polynomials.

(ii) By Luzin's theorem, we select a continuous function h such that

$$\int_a^b |g(t) - h(t)| dt < \epsilon/3.$$

(iii) By Weierstrass' approximation theorem, we select a polynomial $P(t)$ such that

$$\sup_{a \le t \le b} |h(t) - P(t)| < \frac{\epsilon}{3(b-a)}.$$

So that

$$\int_a^b |g(t) - P(t)| dt < 2\epsilon/3.$$

(iv) Finally,

$$\begin{aligned} \Big| \int_a^b g(t) \sin nt \, dt \Big| &\le \Big| \int_a^b (g(t) - P(t)) \sin nt \, dt \Big| + \Big| \int_a^b P(t) \sin nt \, dt \Big| \\ &\le \int_a^b |g(t) - P(t)| dt + \Big| \int_a^b P(t) \sin nt \, dt \Big| \end{aligned}$$

By step (i) there is an n_0 such that for all $n \ge n_0$ the second piece in the last expression is less than $\epsilon/3$ and together with step (iii) we get that

$$\Big| \int_a^b g(t) \sin nt \, dt \Big| < \epsilon, \quad n \ge n_0.$$

The proof is thereby completed.

The issue with continuous functions Question: If f (x) is *merely* assumed to be continuous does the above result hold ? This was believed to be so by several mathematicians including Dirichlet until *Paul Du Bois Reymond* after several abortive attempts at proving it, produced a counter example in 1875 ! Using ideas from set topology one can show that a *majority of continuous functions* display such errant behaviour. The simplified proof given by *Stephan Banach* is available in most texts. We shall discuss this later in the course.

An example: Let us look at a simple case where this theorem is applicable. Consider the function

$$f(x) = |x|, \quad |x| \le \pi$$

extended as a 2π periodic function. Sketch the graph of the function and check that the function is Lipschitz. Since the function is an even function,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

Exercise: Determine all the Fourier coefficients using formulas (1.4)-(1.5) and deduce that

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4\cos(2k-1)x}{\pi(2k-1)^2}$$

What do you get when $x = 0$?