Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

29 Properties of Legendre polynomials

We record here the three defining properties of $P_k(x)$. Note that $k = 0, 1, 2, \ldots$.

1. $P_k(x)$ is a polynomial of degree k
2. $P_k(x)$ satisfies the Legendre Equation:

$$(1-x^2)P_k''(x) - 2xP_k'(x) + k(k+1)P_k(x) = 0$$

3. $P_k(1) = 1.$

It is clear that $P_k(x)$ is an odd function if k is odd and an even function if k is even. Also

$$P_0(x) = 1$$

Theorem (Orthogonality Properties of Legendre Polynomials): If $k \neq l$ then $P_k(x)$ and $P_l(x)$ are orthogonal in the following sense:

$$\int_{-1}^{1} P_k(x)P_l(x)dx = 0.$$

To prove this we begin with the differential equations

$$(1-x^2)P_k'' - 2xP_k' + k(k+1)P_k = 0 \tag{5.8}$$

and

$$(1-x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0 \tag{5.9}$$

We shall write these equations can be written in a more following convenient form known as *self-adjoint form*.

$$\frac{d}{dx}\Big((1-x^2)P_k'\Big) + k(k+1)P_k = 0 \tag{5.10}$$

and

$$\frac{d}{dx}\Big((1-x^2)P_l'\Big) + l(l+1)P_l = 0 \tag{5.11}$$

Multiply (5.10) by P_l, (5.11) by P_k, subtract and integrate over $[-1, 1]$. Integration by parts would confirm that

$$(k(k+1) - l(l+1))\int_{-1}^{1} P_k(x)P_l(x)dx = 0.$$

Since $k \neq l$ and are non-negative integers, the factor $k(k+1) - l(l+1) \neq 0$ and the proof is complete. Exercise: Explain what happens if k and l are not non-negative integers.

Theorem The Legendre polynomials $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots\}$ form a complete orthogonal system. The completeness follows immediately from the Weierstrass approximation theorem.

Theorem (Fundamental Orthogonality Lemma): Suppose V is a vector space endowed with inner product with respect to which $\{v_0, v_1, v_2, \dots\}$ and $\{w_0, w_1, w_2, \dots\}$ are two orthogonal systems of non-zero vectors. Further assume that

$$\text{span}(v_0, v_1, \dots, v_k) = \text{span}(w_0, w_1, \dots, w_k), \quad \text{for every } k = 0, 1, 2 \dots$$

Then, for certain scalars c_k $(k = 0, 1, 2 \dots)$,

$$v_k = c_k w_k, \quad \text{for every } k = 0, 1, 2 \dots$$

Proof is an Exercise. First think of what happens in ordinary Euclidean spaces like $\mathbb{R}^n$. Geometrical considerations suggests the proof. It is an immediate corollary from this result that if $1, x, x^2, \dots$ is subjected to the Gram-Schmidt process with respect to the usual inner product in $L^2[-1, 1]$ the result is the sequence

$$\frac{P_0(x)}{\|P_0(x)\|}, \frac{P_1(x)}{\|P_1(x)\|}, \frac{P_2(x)}{\|P_2(x)\|}, \dots$$

Exercise: Consider the sequence of polynomials

$$Q_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Show that $Q_n(x)$ has degree n for every n. Further show that the sequence is orthogonal with respect to weight 1 namely,

$$\int_{-1}^{1} Q_n(x) Q_m(x) dx = 0, \quad m \neq n.$$

From this infer the following result:

$$P_n(x) = c_n Q_n(x), \quad \text{for every } n = 0, 1, 2 \dots$$

for a certain sequence of constants $\{c_n\}$.

Rodrigues' Formula: Compute the constants c_n in the last slide by evaluating $Q_n(1)$. Deduce the following formula due to *Olinde Rodrigues*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n \tag{5.12}$$

For more information on the work of Rodrigues see the Book Review by *W. P. Johnson*, in the American Math. Monthly, Volume 114, Oct 2007, 752-758.

Exercises:

1. Compute $\int_{-1}^{1} (P_n(x))^2 dx$
2. Show that $\int_{-1}^{1} (1 - x^2)(P_n'(x))^2 dx = 2n(n+1)/(2n+1)$. Hint: Multiply the Diff. Eqn by P_n and integrate by parts.
3. Use Rodrigues formula to prove that the Legendre polynomial of degree n has precisely n distinct roots in the open interval $(-1, 1)$. Use Rolle's theorem. Note: The roots were used by *Gauss in 1814 in his famous quadrature formula. See the discussion on pp. 56-69 of S. Chandasekhar, Radiative transfer, Dover Publications, Inc., New York, 1960.*
4. Show that the Legendre polynomials satisfy the three term recursion formula

$$(n+1)P_{n+1} - x(2n+1)P_n + nP_{n-1} = 0.$$