Fourier Analysis and its Applications

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28 Fourier Bessel series

In the first of these integrals we perform an integration by parts:

$$-\int_0^1 x\phi_k'(x)\phi_j'(x)dx + \zeta_j^2\int_0^1 x\phi_j(x)\phi_k(x)dx - p^2\int_0^1 \phi_j(x)\phi_k(x)\frac{dx}{x} = 0$$

Explain why the boundary terms drop out. Now switch the roles of j, k and subtract the two equations and we get:

$$(\zeta_j^2 - \zeta_k^2)\int_0^1 x\phi_j(x)\phi_k(x)dx = 0$$

Since ζ_j, ζ_k are distinct zeros we get at once the orthogonality of ϕ_j and ϕ_k with respect to the weight xdx.

Now take up the case $p = 0$ where the equation for $\phi_j(x)$ simplifies to:

$$\Big(x\phi_j'(x)\Big)' + x\zeta_j^2\phi_j(x) = 0.$$

Multiply by $\phi_k(x)$ and integrate over $[0, 1]$. The rest of the argument is similar to the previous case.

Let us now take up the the question of evaluating

$$\int_0^1 x(J_p(\zeta_j x))^2 dx$$

We begin with $\phi_u(x) = J_p(ux)$ which satisfies the differential equation

$$x^2\phi_u''(x) + x\phi_u'(x) + (x^2u^2 - p^2)\phi_u = 0$$

At this juncture, u need not be a zero of $J_p(x)$. Multiply by $x^{-1}\phi_v$ and integrate over $[0, 1]$:

$$\int_0^1 x\phi_v(x)\phi_u''(x)dx + \int_0^1 \phi_v(x)\phi_u'(x)dx + \int_0^1 (x^2u^2 - p^2)\phi_u(x)\phi_v(x)\frac{dx}{x} = 0. \tag{5.17}$$

If we switch the roles of u and v and subtract the term involving p^2 will drop out and so we shall supress this term altogether. Integration by parts in the first integral gives a sum of three terms:

$$-\int_0^1 x\phi_v'(x)\phi_u'(x)dx - \int_0^1 \phi_v(x)\phi_u'(x)dx + x\phi_v(x)\phi_u'(x)\Big|_0^1$$

The middle integrals in the last two displays cancel out. The first term in the above display would drop out when we switch the roles of u, v and subtract. Next we look at the boundary terms:

$$x\phi_v(x)\phi_u'(x)\Big|_0^1 = uJ_p(v)J_p'(u)$$

Finally the result of switching the roles of u, v in (5.17) and subtract we would be left with

$$(u^2 - v^2)\int_0^1 x\phi_u(x)\phi_v(x)dx = vJ_p(u)J_p'(v) - uJ_p(v)J_p'(u) \tag{5.18}$$

So far u, v were arbitrary. Now let u *be a zero of* $J_p(x)$ and let $v \to u$. First term on RHS of (5.18) drops out. We get in the limit

$$\int_0^1 x\phi_u^2(x)dx = \lim_{v\to u}\frac{-uJ_p(v)J_p'(u)}{u^2 - v^2} = \frac{1}{2}(J_p'(u))^2$$

using L'Hospital's rule. Proof of Lommel's formula is complete.

Uniqueness of the solution obtained We shall now demonstrate that the solution to the problem of vibrating membrane is unique. Thus the solution we have obtained completely settles the matter. Suppose z_1 and z_2 are two solutions satisfying the same initial conditions and boundary conditions (namely vanishing along the boundary of the membrane $r = 1$). Then the difference $Z = z_1 - z_2$ also satisfies the PDE with zero initial and boundary conditions:

$$\begin{aligned} c^2 \Delta Z - \frac{\partial^2 Z}{\partial t^2} &= 0, \\ Z(x, y, 0) = 0, \; Z_t(x, y, 0) &= 0, \\ Z(\cos\theta, \sin\theta, t) &= 0. \end{aligned}$$

The Energy Method The idea of proof is well-known under the name of *energy method.* The method is frequently employed in the analysis of PDEs. Multiply the differential equation by Z_t and integrate over the disc $D \,:\, x^2 + y^2 \leq 1$. Integration by parts gives (how?)

$$\frac{d}{dt} \iint_D Z_t^2 dxdy + c^2 \iint_D (Z_{xt} Z_x + Z_{yt} Z_y) dxdy = 0.$$

From this we infer that the energy

$$E(t) = \iint_D (c^2 Z_x^2 + c^2 Z_y^2 + Z_t^2) dxdy.$$

is constant in time. Since $E(0) = 0$ we see that Z is a constant and so is zero. The proof is complete.

11. Imitate the energy method to show that the twice continuously differentiable solution to the initial-boundary value problem

$$u_t = \Delta u, \; u(x, y, 0) = f(x, y), \; u(\cos\theta, \sin\theta, t) = 0$$

for the heat equation is unique. Hint: Here the energy function is monotone decreasing in time.

12. Show that the twice continuously differentiable solution to the boundary value problem

$$\begin{aligned} \Delta u = 0, &\qquad \text{on } D \\ u(x, y, z) = f(x, y, z), &\qquad \text{on } \partial D \end{aligned}$$

is unique where D is a region in $\mathbb{R}^3$ with a smooth boundary ∂D and f is a function prescribed along ∂D.

For more detailed discussion on these types of wave phenomena see

1. *Courant and Hilbert, Methods of Mathematical Physics, Volume I.* We have already cited this. For the discussion of vibration of a circular plate (which is more involved than the membrane) see pp. 307-308. These involve Bessel's functions of imaginary orders ip ($p > 0$), known as the modified Bessel's functions.

2. *Lord Rayleigh, Theory of Sound, Volume - I.* This is a comprehensive account of the theory of vibrations. See particularly the long and detailed discussion on vibrating plate. This is still the best source on the *Physics of Vibrations.*

3. For other applications such as the *skin effect* see *F. Bowman, Introduction to Bessel's functions, Dover. 1958.*

4. The Bessel functions also appear in optics. The radii of the successive interference fringes due to diffraction from a circular aperture are given in terms of the zeros of Bessel's function.

5. Besides the authoritative treatise of G. N. Watson mentioned earlier, the book of *Byerly* is particularly recommended especially the historical sketch at the end. The book is available on-line.

The Legendre polynomials We now introduce the Legendre polynomials as a complete orthonormal system for $L^2[-1,1]$. Let us consider the *Legendre's equation:*

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0. \tag{5.3}$$

The coefficient $(1-x^2)$ does not vanish at the origin and so origin is not a *singular point.* We seek a solution of (5.3) in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5.4}$$

Differentiating,

$$-2xy'(x) = \sum_{n=0}^{\infty} -2na_n x^n \tag{5.5}$$

and for the term $x^2y''(x)$ we have

$$x^2y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^n \tag{5.6}$$

The term y'' in the ODE is to be dealt with in the following manner:

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

It is important to arrange it so that the exponent of the general term in the series is n. Here it is $n-2$. So we set $n-2=N$ in the series and we get

$$y'' = \sum_{N=0}^{\infty} (N+2)(N+1)a_{N+2}x^N.$$

Changing the dummy index N to n we see that

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{5.7}$$

Consolidating we get

$$(1-x^2)y'' - 2xy' + p(p+1)y =$$
$$\sum_{n=0}^{\infty} x^n \Big((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n \Big) = 0$$

Equating the coefficient of x^n to zero we get the *recurrence relation:*

$$a_{n+2} = \frac{-a_n(p-n)(p+1+n)}{(n+2)(n+1)}, \quad n = 0, 1, 2, \dots$$

Setting $n = 0, 1, 2, \dots$ in succession we find the list of coefficients:

$$a_2 = -a_0 p(p+1)/2!, \ a_3 = -a_1(p-1)(p+2)/3!$$

$$a_4 = \frac{a_0 p(p-2)(p+1)(p+3)}{4!}, \ a_5 = \frac{a_1(p-1)(p-3)(p+2)(p+4)}{5!}$$

The law of formation is clear:

$$a_{2n} = (-1)^n a_0 \frac{p(p-2)\dots(p-2n+2)(p+1)(p+3)\dots(p+2n-1)}{(2n)!}$$

and

$$a_{2n+1} =$$
$$(-1)^n a_1 \frac{(p-1)(p-3)\dots(p-2n+1)(p+2)(p+4)\dots(p+2n-1)}{(2n+1)!}$$

The general solution is thus given by

$$a_0\Big(1 - \frac{p(p+1)x^2}{2!} + \frac{p(p-2)(p+1)(p+3)x^4}{4!} - \dots\Big) +$$
$$a_1\Big(x - \frac{(p-1)(p+2)x^3}{2!} + \frac{(p-1)(p-3)(p+2)(p+4)x^5}{4!} - \dots\Big).$$

The coefficients a_0 and a_1 may be assigned arbitrary values and assigning the values $a_0 = 1, a_1 = 0$ we get one solution and another linearly independent one by setting $a_0 = 0, a_1 = 1$. Both are power series with unit radius of convergence (*Exercise*).

We see that if p is an integer then *exactly* one of the two series terminates and we have a polynomial solution of the Legendre equation. With a suitable normalization that we shall presently specify, these polynomials are called *Legendre Polynomials*. Assume that $p = k$ is an integer and one of the two series described above terminates into a polynomial solution $f(x)$. One can show without much difficulty that $f(1) \neq 0$. Well, suppose not. Set $x = 1$ in the Diff. Eq. to conclude that $f'(1) = 0$ as well. Assume $f^{(n)}(1) = 0$. Differentiate the differential equation n-times using Leibnitz rule and then put $x = 1$. Apply induction on n.

Now that $f(1) \neq 0$ we can normalize our solution $f(x)$ by dividing by $f(1)$ and consider the solution

$$\frac{f(x)}{f(1)}.$$

This special solution is called the $k-$th *Legendre Polynomial*. It is customary to denote this as $P_k(x)$. We record here its three properties. We record here the three defining properties of $P_k(x)$. Note that $k = 0, 1, 2, \ldots$.

1. $P_k(x)$ is a polynomial of degree k

2. $P_k(x)$ satisfies the Legendre Equation:

$$(1 - x^2)P_k''(x) - 2xP_k'(x) + k(k+1)P_k(x) = 0$$

3. $$P_k(1) = 1.$$

It is clear that $P_k(x)$ is an odd function if k is odd and an even function if k is even. Also

$$P_0(x) = 1$$