Fourier Analysis and its Applications

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5. Sturm-Liouville Problems and PDEs. Generalized Fourier expansions

What is special about the orthogonal system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots ?$$

on $L^2[-\pi, \pi]$. For example are there other interesting/useful orthogonal systems?

(1) On $L^2[-1,1]$ we have the sequence of *Legendre polynomials*

$$P_0(x), P_1(x), P_2(x), \ldots,$$

that form a complete orthogonal system.

(2) On $L^2(\mathbb{R})$ we have the family of *Hermite functions* that provide a complete orthogonal system.

General orthogonal systems of functions

Definition Assume that H is a separable Hilbert space and

$$B = \{\phi_1, \phi_2, \ldots, \phi_n, \ldots\} \tag{5.1}$$

is an orthogonal system of *non-zero* elements in H such that linear span of B is dense in H. Then we say that B is a *complete orthogonal system* in H.

The result of chapter 2 can be expressed as saying

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

is a complete orthogonal system in $L^2[-\pi, \pi]$. The literature on orthogonal system of functions is quite vast and here is a reference which is by now quite classic.

G. Sansone, Orthogonal functions, Dover Publications, 1991.

A need for the study of such orthogonal systems of functions originated from several diverse parts of mathematics:

(1) Approximation theory.

(2) Boundary value problems in partial differential equations

(3) Wavelets and image processing. See for example the discussion of *Haar systems* in the book of *Strichartz* cited earlier (pp. 141-148).

(4) Probability theory. See for example *K. R. Parthasarathy, Introduction to probability and measure, Hindustan book agency, 2005.*

(5) Problems in geometric function theory. See for instance *Z. Nehari, Conformal Mappings, Dover, 1975, pp 239-265.* Particularly pp 258-260 on the use of *Tchebycheff polynomials* of the second kind.

Let us consider a Hilbert space H with a complete orthogonal system (5.1). Usually one normalizes the functions ϕ_n and works with the *orthonormal system*

$$\frac{\phi_n}{\|\phi_n\|}, \quad n = 1, 2, 3, \dots \tag{5.2}$$

Given an element $x \in H$ one knows that

$$x = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n + \dots \tag{5.3}$$

for a uniquely determined sequence of coefficients $c_1, c_2, c_3, \dots$, called the *Fourier coefficients* of x with respect to the given orthogonal system. There is a corresponding Bessel's inequality and Parseval formula associated with the series (5.3). The series (5.3) converges to x in the sense that

$$\lim_{n\to\infty}(c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n) = x$$

in the Hilbert space norm. It is clear that the coefficients c_n in the series (5.3) are given by

$$c_n = \frac{\langle x, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} \tag{5.4}$$

We shall return to these general discussions on Hilbert spaces and move on to an application to boundary value problems in partial differential equations.

Vibrations of the circular membrane Let us consider a circular membrane clamped along its rim and set into vibration. The mean position being along the $x-y$ plane and the origin at the center of the membrane. At time t let the displacement from the mean position be $z(x, y, t)$. It is well known (see *Kreyszig, Adv. Engg. Math, 8th edition, pp 616-618*) that z satisfies the wave equation

$$c^2\Big(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\Big) = \frac{\partial^2 z}{\partial t^2} \tag{5.5}$$

where c denotes the wave speed. We seek a special solution of the form

$$z = (A\cos pt + B\sin pt)u(x, y). \tag{5.6}$$

More general solutions can then be determined by superposition. Substituting the Ansatz (5.6) in the PDE we get

$$c^2(A\cos pt + B\sin pt)\Delta u = -p^2(A\cos pt + B\sin pt)u$$

Helmholtz Equation or the Reduced Wave Equation from which we conclude that u must satisfy the equation

$$\Delta u + k^2 u = 0, \tag{5.7}$$

where $k = p/c$. This equation is known as the *Helmholtz's equation* or the *reduced wave equation.*

Exercises:

1. Write the Laplace operator Δ in plane polar coordinates.

2. Write the Laplace operator in $\mathbb{R}^3$ in spherical polar coordinates. Computation gets very UGLY unless you use some cleverness.

The Helmholtz's equation in polar coordinates reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0. \tag{5.8}$$

Separation of variables We now separate the radial and angular variables by setting

$$u(r, \theta) = v(r) \cos n\theta,$$

where n must be an integer owing to 2π periodicity.

Exercise:

3. The function $v(r)$ satisfies
$$r^2 v'' + r v' + (k^2 r^2 - n^2) v = 0 \tag{5.9}$$
Check that this is the Bessel's equation after a rescaling of the variable r.

The indicial for the equation is $\rho^2 - n^2$ and only the positive index gives a solution which is finite at the origin. Since $r = 0$ corresponds to the center of the membrane which always remains at finite distance, the only physically tenable solution is $J_n(kr)$.

The Bessel functions of the first kind Thus we see that our special solution is (recalling $p = ck$),

$$z(x, y, t) = J_n(kr)(A \cos ckt + B \sin ckt) \cos n\theta.$$

Since the membrane is clamped along the rim we see that the solution vanishes along $r = 1$ for all values of θ and t. Thus the following boundary condition must be satisfied:

$$J_n(k) = 0. \tag{5.10}$$

Thus the frequency k must be a zero of the Bessel's function $J_n(x)$ and we shall see later that there is an infinite list of them. The frequencies therefore form a discrete set of values. The most general solutions are then obtained from superpositions whose coefficients are determined via initial conditions and Fourier Analysis. We illustrate this by means of an example where the oscillations are radial.