Fourier Analysis and its Applications

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Radial functions - the Bessel transform: It is an elementary exercise for you to check that if f is an even function then its Fourier transform is also an even function and likewise if $f(x)$ is odd then so is its Fourier transform.

We now look at these properties in a multi-dimensional context namely the Fourier transform of functions of several variables $f(x_1, x_2, \dots, x_n)$. We shall work with the Schwartz class of functions that are infinitely differentiable and decay rapidly at infinity in the sense that for any polynomial $P(x_1, x_2, \dots, x_n)$

$$\sup_{\mathbb{R}^n} \left| P(x_1, x_2, \dots, x_n) \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right| < \infty$$

Fourier transform of functions of several variables: Let us now consider the multi-dimensional Fourier transform of a function $f(x_1, x_2, \dots, x_n)$ which is in the Schwartz class.

The Fourier transform is defined as

$$\widehat{f}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}^n} \exp(-i(x_1\xi_1 + \dots + x_n\xi_n)) f(x_1, \dots, x_n) dx_1 \dots dx_n.$$

Theorem: Assume that the function f depends only on $\sqrt{x_1^2 + \dots + x_n^2}$. Then the Fourier transform $\widehat{f}$ depends only on $\sqrt{\xi_1^2 + \dots + \xi_n^2}$. The important example is of course the Fourier transform of the multi-dimensional Gaussian:

$$f(x_1, x_2, \dots, x_n) = \exp(-x_1^2 - x_2^2 - \dots - x_n^2)$$

which is evidently a radial function. Its Fourier transform is obviously radial.

Let us now turn to the proof of the theorem. Let us write

$$\langle \mathbf{x},\ \xi \rangle = x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n$$

Now let us choose a rotation matrix A such that $A\mathbf{e}_n = \xi/\|\xi\|$. Then

$$\begin{aligned} \widehat{f}(\xi_1, \dots, \xi_n) &= \int_{\mathbb{R}^n} \exp(-i\|\xi\|\langle A\mathbf{e}_n, \mathbf{x}\rangle) f(x_1, \dots, x_n) dx_1 \dots dx_n. \\ &= \int_{\mathbb{R}^n} \exp(-i\|\xi\|\langle \mathbf{e}_n, A^{-1}\mathbf{x}\rangle) f(x_1, \dots, x_n) dx_1 \dots dx_n. \end{aligned}$$

Since $\langle A\mathbf{e}_n, \mathbf{x}\rangle = \langle \mathbf{e}_n, A^T\mathbf{x}\rangle$ and $A^T = A^{-1}$. We now perform the change of variables $\mathbf{x} = A\mathbf{y}$. Since A is orthogonal, its determinant is ± 1 and applying the change of variable formula

$$\widehat{f}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}^n} \exp(-i\|\xi\| y_n) f(y_1, \dots, y_n) dy_1 \dots dy_n.$$

We should now write it in polar coordinates. But we shall do this for $n = 2$ and $n = 3$ only. Write $f(x_1, x_2, \dots, x_n) = F(r)$ where $r = \sqrt{x_1^2 + \dots + x_n^2}$.
For $n = 3$, $y_3 = r\cos\phi$ and

$$\widehat{f}(\xi_1, \xi_2, \xi_3) = \int_0^\infty F(r) r^2 dr \int_0^\pi \exp(-ir\|\xi\|\cos\phi) \sin\phi d\phi \int_0^{2\pi} d\theta$$

Writing $\cos\phi = s$, we get

$$\widehat{f}(\xi_1,\xi_2,\xi_3) = 2\pi\int_0^\infty F(r)r^2 dr \int_{-1}^1 e^{-irs\|\xi\|}ds = \frac{4\pi}{\|\xi\|}\int_0^\infty F(r) r \sin\|\xi\| r dr$$

For the case $n = 2$ we get

$$\widehat{f}(\xi_1,\xi_2) = \int_0^\infty rF(r)dr\int_{-\pi}^{\pi}\cos(r\|\xi\|\sin\theta)d\theta$$

Which is a *Bessel transform*

$$\widehat{f}(\xi_1,\xi_2) = 2\pi\int_0^\infty rF(r)J_0(r\|\xi\|)dr$$

More generally in even space dimensions it reduces to a Bessel transform and in odd dimension a "sine transform".

The appearence of the Bessel transform is significant in diffraction problems through circular apertures.

The Wave Equation Let us begin with the one dimensional wave equation

$$u_{tt} - u_{xx} = 0. \tag{4.19}$$

in the half plane $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ with intial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{4.20}$$

For simplicity let us assume that $f(x)$ and $g(x)$ are in the Schwartz class. Let us take the Fourier transform with respect to the $x-$variable:

$$\widehat{u_t} = \int_{\mathbb{R}}\frac{\partial u}{\partial t}e^{-i\xi x}dx = \frac{\partial}{\partial t}\int_{\mathbb{R}}u(x,t)e^{-i\xi x}dx = \frac{\partial\widehat{u}}{\partial t}(\xi,t)$$

Likewise,

$$\widehat{u_{tt}} = \frac{\partial^2\widehat{u}}{\partial t^2}(\xi,t)$$

So Fourier transforming the wave equation with respect to the space variable gives us the ODE (regarding ξ as a parameter):

$$\frac{d^2\widehat{u}}{dt^2} + \xi^2\widehat{u} = 0, \tag{4.21}$$

with initial conditions

$$\widehat{u}(\xi,0) = \widehat{f}(\xi), \quad \widehat{u}_t(\xi,0) = \widehat{g}(\xi) \tag{4.22}$$

The solution is immediate

$$\widehat{u}(\xi,t) = (\cos t\xi)\widehat{f}(\xi) + \Big(\frac{\sin t\xi}{\xi}\Big)\widehat{g}(\xi) \tag{4.23}$$

One can appeal to the convolution theorem and obtain an integral representation (*The D'Alembert's formula*) but we shall do something more interesting instead. Let us now multiply the wave equation by u_t and integrate over space:

$$\int_{\mathbb{R}} u_t u_{tt}dx - \int_{\mathbb{R}} u_t u_{xx}dx = 0.$$

Integrate by parts in the second integral and we get

$$\frac{d}{dt}\Big(\int_{\mathbb{R}} u_t^2 dx + \int_{\mathbb{R}} u_x^2 dx\Big) = 0 \tag{4.24}$$

where we have assumed that the boundary terms go since f and g are in the Schwartz class.

Exercise: Consult any book for the statement and proof of the D'Alembert's formula and check the formula using Fourier inversion theorem on (4.23) for the case $f = 0$.

The law of conservation of energy. Kinetic and potential energies: From (4.24) we conclude that the expression

$$E(t) = \int_{\mathbb{R}} u_t^2 dx + \int_{\mathbb{R}} u_x^2 dx \tag{4.25}$$

is actually independent of time which is the mathematical expression for the *law of conservation of energy*. Now let us use the Parseval formula to rewrite (4.25) as

$$\int_{\mathbb{R}} (\widehat{u}_t(\xi,t))^2 d\xi + \int_{\mathbb{R}} \xi^2(\widehat{u}(\xi,t))^2 d\xi = 2\pi E \tag{4.26}$$

Since the energy is independent of time we can put $t = 0$ and obtain

$$\int_{\mathbb{R}} (\widehat{g}(\xi))^2 d\xi + \int_{\mathbb{R}} \xi^2(\widehat{f}(\xi))^2 d\xi = 2\pi E \tag{4.27}$$

Now the kinetic energy $K(t)$ is given by

$$K(t) = \int_{\mathbb{R}} u_t^2 dx = \frac{1}{2\pi}\int_{\mathbb{R}} (\widehat{u}_t(\xi,t))^2 d\xi \tag{4.28}$$

We use the explicit formula (4.23) namely,

$$\widehat{u}(\xi,t) = (\cos t\xi)\widehat{f}(\xi) + \Big(\frac{\sin t\xi}{\xi}\Big)\widehat{g}(\xi), \tag{4.23}$$

in the kinetic energy expression (4.28) and we get

$$K(t) = \int_{\mathbb{R}} u_t^2 dx = \frac{1}{2\pi}\int_{\mathbb{R}} (\widehat{g}(\xi)\cos t\xi - \xi\widehat{f}(\xi)\sin t\xi)^2 d\xi \tag{4.29}$$

Principle of Equipartioning of Energy for the Wave Equation: Now we expand $(\widehat{g}(\xi)\cos t\xi - \widehat{f}(\xi)\sin t\xi)^2$ and express $\cos^2 t\xi, \sin^2 t\xi$ and $2\cos t\xi \sin t\xi$ in terms of $\cos 2t\xi$ and $\sin 2t\xi$.

$$K(t) = \frac{1}{4\pi}\int_{\mathbb{R}} \{\widehat{g}(\xi)^2 + \xi^2\widehat{f}(\xi)^2\} d\xi + \dots = \frac{E}{2} + \dots \tag{4.30}$$

The terms which have been left out as dots are a linear combination of

$$\int_{\mathbb{R}} |\xi\widehat{f}(\xi)|^2 \cos 2t\xi d\xi, \quad \int_{\mathbb{R}} |\widehat{g}(\xi)|^2 \cos 2t\xi d\xi, \quad \int_{\mathbb{R}} |\xi\widehat{f}(\xi)\widehat{g}(\xi)| \sin 2t\xi d\xi$$

We now show that each of these individually tend to zero as $t \longrightarrow \infty$. Well, f and g are in the Schwartz class and so the squares $|\widehat{g}(\xi)|^2$, $|\xi\widehat{g}(\xi)|^2$ as well as the product $|\xi\widehat{f}(\xi)\widehat{g}(\xi)|$ are all in $L^1(\mathbb{R})$ and so we may appeal to the Riemann Lebesgue lemma. Thus we conclude that when time goes to infinity the kinetic energy tends to exactly one half the total energy which means that the potential energy also converges to $E/2$. In other words we see the phenomen of *Equipartitioning of energy. Reference: R. Strichartz, A guide to distribution theory and Fourier transforms, CRC Press LLC, 1994.*