Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

23 Exercises on Fourier transform

1. Use Parseval formula to compute the integral

$$\int_{-\infty}^{\infty} \frac{4\sin^2 \xi d\xi}{\xi^2}$$

Use the fact that $\|f\|^2 = (2\pi)^{-1}\|\widehat{f}\|^2$ where, the norms are all L^2 norms.

2. Use the convolution theorem to determine the convolution $f_s * f_t$ where $f_s(\xi)$ is the Cauchy distribution:

$$f_s(\xi) = \frac{1}{\pi}\frac{s}{\xi^2 + s^2}$$

The Cauchy distribution $f_s(\xi)$ is the Fourier transform of some known function $F_s(x)$. By the inversion theorem can you easily write $\widehat{f_s}$?

3. Compute the Fourier transform of $(\cosh ax)^{-1}$ where a is real positive.

Well, the problem amounts to computing the integral

$$I(\xi) = \int_{-\infty}^{\infty} \frac{\cos \xi x \ dx}{\cosh ax}$$

You could use complex analysis to do this. Usually when you have hyperbolic functions in the denominator it is convenient to use rectangular contours. Select a rectangular contour with vertices $-R, R, R+it$ and $-R+it$ where t has to be selected appropriately so that the integral over the top edge is a multiple of the integral you want. You need to show that the contributions from the vertical sides go to zero.

There is a simple pole inside the contour at the point $z = i\pi/2a$.

4. Suppose $f, g \in L^1(\mathbb{R})$. Show that the convolution $f * g$ is again in $L^1(\mathbb{R})$ and further

$$\|f * g\| \leq \|f\|\|g\|$$

where the norms are all L^1 norms. Can something more be said if the functions are non-negative?

Remark: The above is a special case of a general result called *Young's convolution theorem*. The latter states that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ such that $p^{-1} + q^{-1} = 1 + r^{-1}$ for some $r \geq 1$ then $f * g \in L^r(\mathbb{R})$. Here $p = q = r = 1$. Note that if $p = q = 2$ then $r = \infty$ and the theorem says that the convolution of two functions in $L^2(\mathbb{R})$ is a bounded measurable functions. Prove this theorem. First assume that $f, g \in \mathcal{S}$ and show that for all $x \in \mathbb{R}$:

$$|(f * g)(x)| \leq \|f\|_2\|g\|_2.$$

Can you now use an approximation argument such as density if $\mathcal{S}$ in $L^2(\mathbb{R})$?

For fixed $x \in \mathbb{R}$, look at the map $(f, g) \mapsto (f * g)(x)$ from $\mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$. Can you prove that this map is continuous with respect to the L^2 norm?

5. Let $f(t)$ be the characteristic function of $[-1, 1]$. Find explicitly $f * f$ and show that it is continuous. It is a fact that the convolution of two characteristic functions of bounded measurable sets is continuous. Show that $f * f * f$ is of class C^1. In fact if f_j (for $j = 1, 2, \ldots, k$) is a set of characteristic functions of intervals I_j then $f_1 * f_2 * \cdots * f_k$ is of class C^{k-2}.

6. Observe that $(\sin x)/x$ is not in $L^1(\mathbb{R})$ but it is in $L^2(\mathbb{R})$. Find its Fourier transform by proceeding formally and then resorting to the $e^{-\epsilon x^2}$ trick. Does the result match with what you would expect if you blindly apply the inversion theorem?

7. Suppose f and g are two functions in $L^1(\mathbb{R})$ such that f vanishes outside an interval I and g vanishes outside an interval J then prove that $f * g$ vanishes outside the interval $I + J$. In other words

$$\text{supp } (f * g) \subset \text{supp } f + \text{supp } g$$

Of course one would wonder if equality holds. In fact the inclusion can be strict. What is true is the following result known as the *Titchmarsh convolution theorem:*

$$\text{ch. supp } (f * g) = \text{ch. supp } f + \text{ch. supp } g$$

where ch. supp$f$ denotes the convex hull of the support of f.

8. Let $\phi \in \mathcal{S}$ and real valued. Define

$$\Phi(x) = \sum_{n=-\infty}^{\infty} \phi(x + 2\pi n)$$

Evidently $\Phi(x)$ is periodic with period 2π. Prove that $\Phi(x)$ is smooth.

Show that the Fourier coefficients of $\Phi(x)$ is given by

$$a_k = \sum_{n=-\infty}^{\infty} \int_{-\pi+2\pi n}^{\pi+2\pi n} \phi(u) \cos kudu.$$

Justify the exchange of summation and integrals. So the Fourier coefficients are expressed in terms of the real and imaginary parts of $\widehat{\phi}(k)$. What do you deduce from the basic convergence theorem of chapter 1? The result is known as the *Poisson summation formula.*