Fourier Analysis and its Applications

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21 Heat equation. Heat kernel

The Heat equation again. The heat kernel. Let us now solve the initial value problem for the heat equation in the half-plane

$$u_t - u_{xx} = 0, \quad u(x,0) = f(x).$$

Let us assume to begin with $f(x) \in \mathcal{S}$ and compute the Fourier transform with respect to x:

$$\frac{d}{dt}(\widehat{u}(\xi,t)) + \xi^2\widehat{u} = 0, \quad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

This is an ODE in $\widehat{u}$ where ξ is regarded as a parameter.

$$\widehat{u}(\xi,t) = C\exp(-t\xi^2)$$

Putting in $t = 0$ we see that $C = \widehat{f}(\xi)$. Thus

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\exp(-t\xi^2)$$

Observe that $\exp(-t\xi^2)$ is the Fourier transform with respect to x, of the function

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/4t)$$

Appealing to the convolution theorem,

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\widehat{G}(\xi,t) = \widehat{G * f}(\xi,t)$$

Thus we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(s)\exp(-(x-s)^2/4t)\,ds$$

The function $G(x,t)$ called *the heat kernel,*plays a crucial role in *Probability theory.* The formula was derived assuming that the initial data $f(x)$ is in $\mathcal{S}$. However it makes perfect sense even if $f(x)$ is of exponential type ! Exercises:

29. Solve the heat equation $u_t - u_{xx} = 0$ with initial condition $u(x,0) = x^2$.

30. Solve the heat equation $u_t - u_{xx} = 0$ with initial condition $u(x,0) = \cos(ax)$. What about the solution with initial condition $\sin(ax)$.

31. Suppose the initial condition is a continuous function that is positive at say on $(-1,1)$ but zero outside $[-1,1]$ then the solution is positive at all points $u(x,t)$ no matter how large x is and how small $t > 0$ is. *Thus the effect of initial heat distribution in* $[-1,1]$ *is instantaneously propagated throughout space. Is this physically tenable?* Philosophical Question: How is it that the equation nevertheless is used to explain physical phenomena??

Airy's Function: Airy studied the function that bears his name in the course of his investigations on the intensity of light in the neighborhood of a caustic (See G. N. Watson's treatise, p. 188). The work dates back to 1838. Before commencing on the discussion of Airy's function, here is a pointer to the interesting life of *Sir George Biddell Airy:*
Sir George Biddell Airy, K. C. B, Ed., Wilfred Airy, CUP, 1896. Another interesting account I had read long ago was by *Patrick Moore* but I am unable to locate it at the moment.

Airy's equation is the ODE

$$y''(x) - \frac{1}{3}xy = 0. \tag{4.15}$$

G. N. Watson, A treatise on the theory of Bessel functions, Second Edition, CUP, 1958

Some preliminaries on conditionally convergent integrals: Before taking up Airy's equation let us recall a few useful methods of dealing with conditionally convergent intgerals. The most basic integral is

$$\int_0^\infty \frac{\sin x dx}{x} \tag{4.16}$$

which is known to converge to $\pi/2$. Let us split the integral into two parts

$$\int_0^1 \frac{\sin x dx}{x} + \int_1^\infty \frac{\sin x dx}{x}$$

The first piece is the integral of a continuous function on a closed bounded interval and so it exists. We must now turn to the second integral. Let us denote by I the second integral namely

$$I = \int_1^\infty \frac{\sin x dx}{x} = \lim_{R\to\infty}\int_1^R \frac{\sin x dx}{x} = \lim_{R\to\infty}\int_1^R -\frac{d}{dx}(\cos x)\frac{dx}{x}$$

Integrating by parts we get

$$\begin{aligned} I &= \lim_{R\to\infty}\Big(\int_1^R -\frac{\cos x dx}{x^2} + \cos 1 - \frac{\cos R}{R}\Big) \\ &= \cos 1 - \int_1^\infty \frac{\cos x dx}{x^2} \end{aligned}$$

The last displayed integral evidently converges absolutely !

Let us now take another example which is important since the Airy's function will be an integral of a similar but more complicated form. Consider the *Fresnel integral:*

$$\int_0^\infty \cos x^2 dx$$

Again we need to discuss convergence of only the integral from $[1, \infty)$ which we call J. We make the change of variables $x^2 = u$ and we get

$$J = \int_1^\infty \frac{\cos u du}{2\sqrt{u}}$$

Now the same idea of integrating by parts would show that J converges. Exercise: Show that the integrals I and J do not converge absolutely.

Integral representation of Airy's function: We would like to subject Airy's equation to Fourier transform. But the main question is whether the equation has solutions which are amenable to Fourier transforms? Since Airy's equation (after a trivial change of variables $x \mapsto -x$)

$$y''(x) + \frac{1}{3}xy = 0. \tag{4.15$'$}$$

does not contain the y' term, the Wronskian of any two linearly independent solutions is a non-zero constant. So if one solution decays at infinity along with its derivative then the second linearly independent solution must grow rapidly at infinity. So *at most one solution* (upto scalar multiples) can be subjected to Fourier transforms. We leave aside the question of whether there is such a solution and proceed formally. Taking the Fourier transform of (4.15) we get the ODE:

$$\xi^2\widehat{y} - \frac{1}{3i}\frac{d\widehat{y}}{d\xi} = 0 \tag{4.16}$$

Integrating this linear ODE we get

$$\widehat{y} = \exp(i\xi^3)$$

Now we use the Fourier inversion theorem and obtain, ignoring multiplicative constants,

$$y(x) = \int_{\mathbb{R}} \exp(ix\xi + i\xi^3)d\xi \tag{4.17}$$

which can be written as

$$y(x) = \int_{\mathbb{R}} \cos(x\xi + \xi^3)d\xi \tag{4.18}$$

The problem is that the integral (4.18) is a conditionally convergent integral and the steps leading to (4.18) are suspect. We could however directly try and verify that the integral (4.18) is a solution of the ODE but that too is problematic since differentiation under the integral sign is not easy to justify. Let us now make the change of variables $x\xi + \xi^3 = s$. The function $x\xi + \xi^3$ is monotone increasing on say $[r, \infty)$ and on this interval the change of variables is licit. The integral transforms into

$$y(x) = \int_R^\infty \frac{\cos s ds}{x + 3\xi^2} \tag{4.19}$$

where ξ is a function of s. Clearly $\xi(s) \to \infty$ as $s \to \infty$. Let us integrate by parts the integral (4.19) and we are led to discussing the convergence of

$$6\int_R^\infty \frac{\xi(s)(\sin s)\xi'(s)ds}{(x+3\xi^2)^2} = 6\int_R^\infty \frac{\xi(s)(\sin s)ds}{(x+3\xi^2)^3} \tag{4.20}$$

Exercise: Write down the boundary terms arising from integration by parts.