Fourier Analysis and its Applications

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20 Plancherel's theorem

Theorem (Riemann-Lebesgue lemma for Fourier transforms): Suppose $f \in L^1(\mathbb{R})$, then $\widehat{f}(\xi)$ tends to zero as $\xi \longrightarrow \pm\infty$.

Proof: First imitate the above argument to show that if f is continuous on $[-A, A]$ then

$$\lim_{\xi \to \pm\infty} \int_{-A}^{A} f(x) e^{-ix\xi} dx = 0.$$

Call the integral I and set $x = y + \frac{\pi}{\xi}$ and proceed as we did before. Use Luzin's theorem to prove it for all $f \in L^1[-A, A]$. Now suppose $f \in L^1(\mathbb{R})$. Let $\epsilon > 0$ be arbitrary. Select $A > 0$ such that

$$\int_{\mathbb{R}-[-A,A]} |f(x)| dx < \epsilon/2.$$

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \right| &\leq \left| \int_{\mathbb{R}-[-A,A]} f(x) e^{-ix\xi} dx \right| + \int_{[-A,A]} f(x) e^{-ix\xi} dx \right| \\
&\leq \int_{\mathbb{R}-[-A,A]} |f(x)| dx + \left| \int_{[-A,A]} f(x) e^{-ix\xi} dx \right| \\
&\leq \epsilon/2 + \left| \int_{[-A,A]} f(x) e^{-ix\xi} dx \right|
\end{aligned}$$

Since we know that $\displaystyle\int_{[-A,A]} f(x) e^{-ix\xi} dx \longrightarrow 0, \quad \xi \to \pm\infty$ there is a $\xi_0 > 0$ such that for all $|\xi| > \xi_0$, we have $\displaystyle\left| \int_{[-A,A]} f(x) e^{-ix\xi} dx \right| < \epsilon/2$ and accordingly, $\displaystyle\left| \int_{\mathbb{R}} f(x) e^{-ix\xi} dx \right| < \epsilon$

Estimates in L^2 norm. We would like to get estimates for the Fourier transform in the L^2 norm due to the pleasant feature that $L^2(\mathbb{R})$ is a Hilbert space. However the argument is not so straightforward.

Theorem (The Parseval formula also known as Plancherel's theorem): Suppose $f(t)$ and $g(t)$ are in $\mathcal{S}$ then

$$\int_{-\infty}^{\infty} f(t) \overline{g(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \tag{4.11}$$

Again we need to employ the $\exp(-\epsilon\xi^2)$ trick. First, let us try to prove it directly:

$$\begin{aligned}
RHS &= \int_{\mathbb{R}} d\xi \int_{\mathbb{R}^2} f(x) \overline{g(y)} e^{-i\xi(x-y)} dx dy \\
&= \int_{\mathbb{R}^2} f(x) \overline{g(y)} dx dy \int_{\mathbb{R}} e^{-i\xi(x-y)} d\xi
\end{aligned}$$

This suggests introduction of the $\exp(-\epsilon\xi^2)$. We need the Fourier transform of the Gaussian along the way:

$$\begin{aligned} RHS &= \lim_{\epsilon\to 0+}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\exp(-\epsilon\xi^2)d\xi \\ &= \lim_{\epsilon\to 0+}\frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)\overline{g(y)}dxdy\int_{\mathbb{R}}\exp(-i\xi(x-y))\exp(-\epsilon\xi^2)d\xi \\ &= \lim_{\epsilon\to 0+}\frac{1}{2\sqrt{\pi\epsilon}}\int_{\mathbb{R}} f(x)dx\int_{\mathbb{R}}\overline{g(y)}\exp(-(x-y)^2/4\epsilon)dy \end{aligned}$$

In the inner integral put $y = x + 2\sqrt{\epsilon}z$ and we get

$$RHS = \lim_{\epsilon\to 0+}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x)dx\int_{\mathbb{R}}\overline{g(x+2\sqrt{\epsilon}z)}e^{-z^2}dz$$

Now appealing to the dominated convergence theorem we get the result.

Fourier transform as an operator on $L^2(\mathbb{R})$ Form the Plancherel's theorem it immediately follows taking $f = g$:

Theorem: If $f \in \mathcal{S}$ then

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_2 \tag{4.12}$$

Theorem: The Fourier transform extends as a bounded linear operator on $L^2(\mathbb{R})$ and further it is a linear isomorphism onto $L^2(\mathbb{R})$. To prove this result we use the fact that the space $\mathcal{S}$ is dense in $L^2(\mathbb{R})$. *We shall not prove this here.* Now that we have established (4.12), let us denote by $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ the Fourier transform as an operator on $\mathcal{S}$. It is onto thanks to the inversion theorem. Observe that the map $\mathcal{F}$ being linear, we get

$$\|\mathcal{F}f - \mathcal{F}g\|_2 \leq \sqrt{2\pi}\|f-g\|_2, \quad f, g \in \mathcal{S},$$

and thanks to the inversion theorem,

$$\|\mathcal{F}^{-1}f - \mathcal{F}^{-1}g\|_2 \leq \frac{1}{\sqrt{2\pi}}\|f-g\|_2, \quad f, g \in \mathcal{S}.$$

which shows that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are both uniformly continuous with respect to the L^2 metric. We now prove a general lemma on metric spaces:

Lemma: Suppose X is a complete metric space, Y is a dense subspace of X and $T : Y \longrightarrow Y$ is a uniformly continuous map then T extends continuously as a map $X \longrightarrow X$.

Proof: Let $x \in X$ and (y_n) be a sequence of points of Y converging to X. Then the sequence $T(y_n)$ is Cauchy since T is uniformly continuous. Thus $T(y_n)$ converges to say $Tx \in X$. We show that if (y_n) and (z_n) are two sequences converging to x then the corresponding sequences $T(y_n)$ and $T(z_n)$ both converge to the same limit. To see this interlace the sequences as

$$y_1, z_1, y_2, z_2, y_3, z_3, \dots$$

which evidently converges and the so the corresponding sequence

$$T(y_1), T(z_1), T(y_2), T(z_2), \dots$$

also converges and so its subsequences all converge to the same limit. Thus $T(x)$ is unambiguously defined. The continuity of the extension is an exercise.

Fourier transform as an operator on $L^2(\mathbb{R})$ Using the above result we see that the Fourier transform which hitherto is defined as an operator from $\mathcal{S}$ onto itself extends as a continuous linear map from $L^2(\mathbb{R})$ onto itself and satisfies the estimate

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_2, \quad f \in L^2(\mathbb{R}) \tag{4.13}$$

We now introduce the notion of convolution of two functions:

Definition (Convolution of two functions): Suppose f and g are two absolutely integrable functions on $\mathbb{R}$ their convolution $f * g$ is the function defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

Exercises: Check that $f * g = g * f$ and that $f * g$ is absolutely integrable.

Theorem (The convolution theorem): Suppose $f(t)$ and $g(t)$ are both in $\mathcal{S}$ then so is their convolution $(f * g)(t)$. Further

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{4.14}$$

We shall not prove that the convolution is in $\mathcal{S}$. Observe the analogy with the corresponding result for Laplace transforms that you may have seen in undergraduate courses on ordinary differential equations.

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi}(f * g)(x)dx \\
&= \int_{\mathbb{R}} e^{-ix\xi}dx \int_{\mathbb{R}} f(y)g(x-y)dy \\
&= \int_{\mathbb{R}} f(y)dy \int_{\mathbb{R}} e^{-ix\xi}g(x-y)dx
\end{aligned}$$

Put $x - y = z$ in the inner integral and we get

$$\widehat{f * g}(\xi) = \int_{\mathbb{R}} f(y)dy \int_{\mathbb{R}} e^{-i(y+z)\xi}g(z)dz = \widehat{f}(\xi)\widehat{g}(\xi).$$

The Heat equation again. The heat kernel. Let us now solve the initial value problem for the heat equation in the half-plane

$$u_t - u_{xx} = 0, \quad u(x, 0) = f(x).$$

Let us assume to begin with $f(x) \in \mathcal{S}$ and compute the Fourier transform with respect to x:

$$\frac{d}{dt}(\widehat{u}(\xi, t)) + \xi^2 \widehat{u} = 0, \quad \widehat{u}(\xi, 0) = \widehat{f}(\xi).$$

This is an ODE in $\widehat{u}$ where ξ is regarded as a parameter.

$$\widehat{u}(\xi, t) = C \exp(-t\xi^2)$$

Putting in $t = 0$ we see that $C = \widehat{f}(\xi)$. Thus

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) \exp(-t\xi^2)$$