Fourier Analysis and its Applications

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02 The basic convergence theorem

Remark: We shall need some additional hypothesis on $f(x)$ in order to establish any of these convergence results. By way of an analogy, recall from elementary calculus the problem of expressing a function $f(x)$ as a Taylor series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \tag{1.2}$$

Let $C^\infty(\mathbb{R})$ be the class of infinitely differentiable functions on the real line. We know that even if $f(x) \in C^\infty(\mathbb{R})$, the function $f(x)$ need not admit a representation of the form (1.2). However for a certain distinguished subclass of C^∞ we do have a representation of the form (1.2) which is valid at least on some open interval $(-R, R)$. Further, in this case we easily obtain the formula

$$a_n = \frac{1}{n!} f^{(n)}(0), \quad n = 0, 1, 2, \dots \tag{1.3}$$

Question: What is the analogue of the pair (1.2)-(1.3) in the context of trigonometric series?

In order to obtain a formula for the coefficients in the series (1.1), let us assume that the series appearing in (1.1) is uniformly convergent so that the 2π periodic function $f(x)$ is continuous. Term by term integration of (1.1) gives immediately

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx. \tag{1.4}$$

Multiplying (1.1) by $\cos kx$ (respectively $\sin kx$) and integrating over $[-\pi, \pi]$ gives immediately

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \tag{1.5}$$

Definition: Suppose $f(x)$ is integrable on $[-\pi, \pi]$, we say the trigonometric series (1.1) is a *Fourier series* if the coefficients a_0, a_n, b_n $(n = 1, 2, 3, \dots)$ in (1.1) are given by (1.4)-(1.5).

Not all trigonometric series are Fourier series Note that if the series (1.1) is not sufficiently well-behaved as regards convergence, the validity of the deduction (1.4)-(1.5) is quite suspect !

In fact even if the series (1.1) converges point-wise everywhere to a sum function $f(x)$, the coefficients may not be given by (1.4)-(1.5). One such example is the classic one (Fatou (1906)):

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{\log n}. \tag{1.6}$$

The series (1.6) converges pointwise everywhere and even uniformly on $[\delta, 2\pi - \delta]$ for every $\delta > 0$. The sum $f(x)$ is NOT Lebesgue integrable on $[-\pi, \pi]$. Indeed in 1875 Paul Du Bois Reymond established that if the sum of a trigonometric series is integrable in the Riemann sense then it is a Fourier series (that is its coefficients must be given by (1.4)-(1.5)). The result was extended in 1912 to Lebesgue integrable functions.

See the book of *G. Bachman, L. Narici and E. Beckenstein, Fourier and wavelet analysis, Springer Verlag, 2000.* (pp. 219-220).

The pointwise convergence theorem

From now on we shall only work with Fourier series of $f(x)$ which is Lebesgue integrable on $[-\pi, \pi]$ namely when the coefficients a_0, a_n, b_n are given by (1.4)-(1.5).

Lemma 1.1:

$$1 + 2\cos\theta + 2\cos 2\theta + \cdots + 2\cos n\theta = \frac{\sin(n\theta + \frac{\theta}{2})}{\sin(\frac{\theta}{2})} \tag{1.7}$$

Let us now prove the lemma by recalling a simple identity from trigonometry:

$$2\cos j\theta \sin(\theta/2) = \sin(j + \frac{1}{2})\theta - \sin(j - \frac{1}{2})\theta.$$

Set $j = 1, 2, \ldots, n$ and add. We get

$$2\sum_{j=1}^{n} \cos j\theta \sin(\theta/2) = \sin(n + \frac{1}{2})\theta - \sin(\frac{\theta}{2}).$$

A little re-arrangement gives (1.7).

The Dirichlet Kernel As a preparation for the point-wise convergence theorem we shall obtain an integral expression for the finite sum

$$S_n(f, x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx) \tag{1.8}$$

Using the integrals (1.4)-(1.5) for the coefficients, the right hand side of (1.8) assumes the form:

$$S_n(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\Big(1 + 2\cos(x - t) + 2\cos 2(x - t) + \cdots + 2\cos n(x - t)\Big) dt.$$

We use the lemma that we just proved to get

$$S_n(f, x) = \int_{-\pi}^{\pi} f(t) D_n(x - t) dt. \tag{1.9}$$

where we have denoted by $D_n(\theta)$ the following expression knows as the *Dirichlet kernel*

$$D_n(\theta) = \frac{1}{2\pi} \frac{\sin(n + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})} \tag{1.10}$$

We make two simple observations:

(1) Suppose P and Q are any two periodic function on the real line with period $2c$, then

$$\int_{-c}^{c} P(t)Q(x - t)dt = \int_{-c}^{c} P(x - t)Q(t)dt \tag{1.11}$$

Hint: Put $x - t = s$ in the left hand side and break the resulting integral into three integrals namely, over the intervals $[-c, x - c], [-c, c]$ and $[c, x + c]$. Now make the change of variables $s \mapsto s - 2c$ in the last of these three integrals.

(2) Integrate both sides of equation (1.7) in Lemma 2 and show that

$$\int_{-\pi}^{\pi} D_n(t) dt = 1. \tag{1.12}$$

Using the simple observation (1.10) we re-write equation (1.9) as

$$S_n(f,x) = \int_{-\pi}^{\pi} f(t)D_n(x-t)dt = \int_{-\pi}^{\pi} f(x-t)D_n(t)dt \tag{1.13}$$

Multiplying (1.12) by $f(x)$ and subtracting from (1.13) we get the important result

$$S_n(f,x) - f(x) = \int_{-\pi}^{\pi} (f(x-t) - f(x))D_n(t)dt. \tag{1.14}$$

In order to prove that *Pointwise Convergence* of the Fourier series to $f(x)$ namely, to prove $S_n(f,x) \longrightarrow f(x)$, we must show that the integral on the right hand side of (1.14) goes to zero as $n \longrightarrow \infty$.

Behaviour of $D_n(t)$ and inadequacy of mere continuity! Let us now try a naive idea that will **NOT** work and we shall try to understand the cause of the failure. It is tempting to take the absolute value of (1.14) and write

$$|S_n(f,x) - f(x)| \leq \int_{-\pi}^{\pi} |f(x-t) - f(x)||D_n(t)|dt. \tag{1.14a}$$

The natural thing would be to appeal to the uniform continutiy and in a neighborhood $t \in [\delta, \delta]$ we estimate

$$|f(x-t) - f(x)| < \epsilon$$

whereas on $|t| > \delta$ all we have is a bound $|f(x-t) - f(x)| \leq M$. To secure that the right hand side of (1.14a) goes to zero we would need that…

$$\int_{-\pi}^{\pi} |D_n(t)|dt$$

decays as $n \to \infty$ but here **our luck has forsaken us !** In fact the truth is:

$$\int_{-\pi}^{\pi} |D_n(t)|dt \sim c\log n, \qquad \text{as } \; n \to \infty$$

for some positive constant c. And we see that there is no way to salvage the argument. Indeed as we know, pointwise convergence **fails** for functions that are merely continuous.