Fourier Analysis and its Applications

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19 Fourier inversion theorem. Riemann Lebesgue lemma

Theorem (The Inversion theorem): Suppose $f(t)$ is a function in $\mathcal{S}$ then the function can be recovered from its Fourier transform via the formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi. \tag{4.10}$$

To prove this first try to substitute in the RHS of (6) the expression for $\widehat{f(\xi)}$ and invert the order of integrals. You will run into the following (hitherto meaningless) integral:

$$\int_{-\infty}^{\infty} \exp(i(t-x)\xi)d\xi$$

Proof of inversion theorem: The $\exp(-\epsilon x^2)$ **trick !** Equipped with this, we now write

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi-\epsilon\xi^2}d\xi$$

Now we put in the definition of $\widehat{f}(\xi)$ and we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(x)e^{i\xi(t-x)-\epsilon\xi^2}dx\Big)d\xi$$

We can now safely invert the order of integral and write

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{i\xi(t-x)-\epsilon\xi^2}d\xi\Big)f(x)dx$$

The inner integral is the Fourier transform of the Gaussian that we have computed !!

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = \lim_{\epsilon\to 0}\sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty} \exp(-(x-t)^2/4\epsilon)f(x)dx$$

Putting $x = t + \sqrt{4\epsilon}s$ we obtain

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{it\xi}d\xi = 2\sqrt{\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(t+\sqrt{4\epsilon}s)e^{-s^2}ds = 2\pi f(t)$$

The proof is complete. Now, a bunch of exercises are collected all of which are amenable to the $\exp(-\epsilon t^2)$ trick.

Exercises:

20. Prove that

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi = 2\pi f(0), \quad f \in \mathcal{S}.$$

If we try to calculate the integral directly then we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$$

If we now try to switch the order of integrals we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi \;=\; \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} e^{-i\xi x}d\xi$$

Note that the inner integral is meaningless as it stands. The switching of the orders of integrals is not valid. To circumvent this difficulty we shall employ the $\exp(-\epsilon\xi^2)$ trick. So let us restart with the observation

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi &= \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} e^{-\epsilon\xi^2}\widehat{f}(\xi)d\xi \\ &= \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} e^{-\epsilon\xi^2}d\xi \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx\end{aligned}$$

Switching the order of integrals we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi \;=\; \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} e^{-\epsilon\xi^2 - ix\xi}d\xi$$

The innermost integral on the RHS is the Fourier transform of the Gaussian that we have computed. Incorporating this we get

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi \;=\; \lim_{\epsilon\to 0+}\frac{\sqrt{\pi}}{\sqrt{\epsilon}}\int_{-\infty}^{\infty} f(x)\exp(-x^2/4\epsilon)dx$$

Put $x = 2y\sqrt{\epsilon}$ we get:

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi \;=\; \lim_{\epsilon\to 0+} 2\sqrt{\pi}\int_{-\infty}^{\infty} f(2y\sqrt{\epsilon})\exp(-y^2)dy$$

The limit passes under the integral sign due to the dominated convergence theorem leading to:

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi \;=\; 2\sqrt{\pi}\int_{-\infty}^{\infty} f(0)\exp(-y^2)dy = 2\pi f(0)$$

21. Compute the Fourier transform of $(\sin at)/t$.

22. Compute the Fourier transform of $f(t) = (t^2 + a^2)^{-1}$ by obtaining a second order ODE.

23. Use the $\exp(-\epsilon t^2)$ trick to obtain a second order ODE for the function

$$\frac{2}{\pi}\int_{1}^{\infty} \frac{(\sin tx)dt}{\sqrt{t^2-1}}, \qquad x > 0.$$

Can you recognize this as a familiar function? Also prove this by taking Laplace transform. This is from *H. Weber, Die Partiellen Differential-Gleichungen der mathematischen Physik, Vol - I, Braunschweig, 1900* based on Riemann's lectures. The formula appears on p. 175. This representation is due to *Mehler (1872) and Sonine (1880). See p. 170 of G. N. Watson's treatise.*

Norm estimates for the Fourier transform We have seen that if $f \in L^1(\mathbb{R})$ then its Fourier transform exists and we have

$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)|dx$$

so that the Fourier transform lands up in $L^\infty(\mathbb{R})$. Further we see that

$$\|\widehat{f}\|_\infty \leq \|f\|_1$$

so that the Fourier transform is a continuous linear map from $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Can we say anything more about the Fourier transform?

The Riemann Lebesgue lemma We now prove a version of the Riemann Lebesgue lemma for Fourier transforms. Before taking it up let us recall the Riemann Lebesgue lemma for Fourier series and give a second proof of the result which dates back to Riemann himself.

Theorem (Riemann Lebesgue lemma revisited): If $f \in L^1[-\pi, \pi]$ then the Fourier coefficients of f decay to zero.

First let us prove it under the assumption that f is continuous. Extend f to the entire real line by declaring

$$f(x) = f(\pi), \quad x \geq \pi, \qquad \text{and } f(x) = f(-\pi), \quad x \leq -\pi.$$

The extension is now bounded uniformly continuous on the entire real line. Let us now take up the integral

$$I = \int_{-\pi}^{\pi} f(x) \sin nx dx.$$