Fourier Analysis and its Applications

Prof. G. K. Srinivasan
Department of Mathematics
Indian Institute of Technology Bombay

18 Hermite's differential equation. Hermite polynomials

The Schwartz space $\mathcal{S}$ of rapidly decreasing functions This is a convenient class of functions introduced by *Laurant Schwartz* in his influential work *Theorié des distributions, Hermann, Paris.* This function space has the advantage that it is a vector space and it closed under differentiation as well as multiplication by polynomials. Besides it also contains enough functions so that any $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

can be approximated arbitrarily closely by functions in $\mathcal{S}$.

This space is particularly well-suited to the study of the Fourier transform. One proves all the results in the context of $\mathcal{S}$ where there are no technical obstructions to differentiation under integral sign or switching order of integrals. To pass on to the more general case one resorts to approximation techniques. The situation is reminiscent of the proof of Parseval formula for Riemann integrable functions.

Definition: The space $\mathcal{S}$ consists of all infinitely differentiable functions $f(t)$ such that for any $m, n \in \mathbb{N}$,

$$t^m \Big(\frac{d^n}{dt^n}\Big) f(t)$$

remains bounded. We immediately see that for any polynomial $P(t)$ and any $a > 0$, the function $P(t)\exp(-at^2)$ lies in $\mathcal{S}$.

12. Show that $(\cosh at)^{-1}$ and $t(\sinh at)^{-1}$ lie in $\mathcal{S}$ for any $a > 0$.

13. One can show using the complex version of *Stirling's approximation formula* that

$$|\Gamma(a + it)|^2$$

lies in $\mathcal{S}$ for $a > 0$. This is an *Optional* exercise.

Properties of $\mathcal{S}$ The space $\mathcal{S}$ is obviously a vector space and it has the following further properties

1. Suppose $f(t)$ and $g(t)$ belong to $\mathcal{S}$ then so does $f(t)g(t)$.

2. It $f(t) \in \mathcal{S}$ then the derivative $f'(t)$ also lies in $\mathcal{S}$.

3. If $f(t) \in \mathcal{S}$ and $P(t)$ is a polynomial then $f(t)P(t) \in \mathcal{S}$.

4. For every $f(t) \in \mathcal{S}$ the integral of $|f(t)|$ over $\mathbb{R}$ exists and so for every $f(t) \in \mathcal{S}$ the Fourier transform is defined.

We shall now show that if $f(t) \in \mathcal{S}$ then $\widehat{f}(\xi)$ also lies in $\mathcal{S}$. Thus the Fourier transform maps $\mathcal{S}$ into itself as a linear transformation. Can you point out some of its eigen-values by looking at the list of Fourier transforms we have calculated?

Theorem: Suppose $f(t) \in \mathcal{S}$ then $\widehat{f}(\xi) \in \mathcal{S}$.

Proof: Differentiating under the integral is permissible since $f(t)$ decays rapidly. Thus

$$\left(\frac{d}{d\xi}\right)^n \widehat{f}(\xi) = (-i)^n \int_{-\infty}^{\infty} e^{-it\xi}(t^n f(t))dt$$

Rewrite the integrand as $\left\{e^{-it\xi}t^n f(t)(1+t^2)\right\}(1+t^2)^{-1}$. The bracketed term is bounded in absolute value. Multiplying the integral by $(-i\xi)^k$ we see

$$(-i\xi)^k\left(\frac{d}{id\xi}\right)^n \widehat{f}(\xi) = (-1)^n \int_{-\infty}^{\infty} \left(\frac{d}{dt}\right)^k e^{-it\xi}(t^n f(t))dt$$

Integrating by parts and using $D^k(t^n f(t)) \in \mathcal{S}$ we conclude $\xi^k D^n \widehat{f}(\xi)$ is bounded for every $n, k \in \mathbb{N}$. Since k is arbitrary, the result is proved.

Differentiation and multiplication:

Theorem Suppose $f(t) \in \mathcal{S}$ then

$$\begin{aligned} \widehat{\left(\frac{d}{idt}\right)(f(t))} &= \xi \widehat{f}(\xi) \\ \widehat{tf(t)}(\xi) &= \left(-\frac{d}{id\xi}\right)\left(\widehat{f}(\xi)\right) \end{aligned}$$

To prove the first part integrate by parts. To prove the second part, differentiate under the integral with respect to ξ:

$$\frac{d}{d\xi}\left(\widehat{f}(\xi)\right) = \frac{d}{d\xi}\int_{-\infty}^{\infty} e^{-it\xi}f(t)dt = \int_{-\infty}^{\infty} -ite^{-it\xi}f(t)dt$$

Divide by $-i$ and we get the second formula.

Hermite's ODE and Hermite polynomials again !!

14. Transform the *Hermite's differential equation* $y'' - 2xy' + 2\lambda y = 0$ by the substitution $y \exp(-x^2/2) = u$. Ans: $u'' - x^2 u + (2\lambda + 1)u = 0$.

15. Show that the equation $y'' - 2xy' + 2\lambda y = 0$ for $\lambda = n \in \mathbb{N}$ has *a polynomial solution of degree exactly* n. Further show that upto scalar multiples, there is *only one* polynomial solution. Any one of these polynomial solutions of degree n is denoted by $H_n(x)$. Unlike the Legendre polynomials there is not universally accepted normalization for $H_n(x)$. The polynomials $\{H_n(x)\}$ are called the *Hermite polynomials.*

16. Rewrite the Hermite's ODE as

$$\left(e^{-x^2}y'\right)' + 2\lambda e^{-x^2}y = 0.$$

Show that if $n \neq m$ then as elements in $L^2(\mathbb{R})$,

$$e^{-x^2/2}H_m(x) \perp e^{-x^2/2}H_n(x)$$

17. Show that if u is a solution for the transformed ODE (see exercise (14))

$$u'' - x^2 u + (2\lambda + 1)u = 0$$

then $\widehat{u}$ is also a solution of the same differential equation. That is, the transformed equation is *invariant* under the Fourier transform.

18. Show that at most one of the solutions of the transformed ODE lies in $\mathcal{S}$. If $\lambda \in \mathbb{N}$ then

$$H_n(x) \exp(-x^2/2)$$

where $H_n(x)$ is the $n-$th Hermite polynomial, lies in $\mathcal{S}$. Hint: The *Abel-Liouville formula.*

19. The Fourier transform is a linear transformation of $\mathcal{S}$ to itself. Show that $H_n(x) \exp(-x^2/2)$ are eigen-vectors of this linear map.