Fourier Analysis and its Applications

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17 Fourier Transform and the Schwartz space

IV - Fourier transforms. The Schwartz space

Recall that in basic ODE theory, where one studies equations with constant coefficients, say

$$y'' + ay' + by = 0$$

one seeks special solutions in the form e^{mx} where m is a root of the characteristic polynomial. More generally one seeks solutions of the form

$$P(x)\exp(mx).$$

Here m is a root of the characteristic equation and $P(x)$ is a polynomial which would be non-constant when the characteristic equation has multiple roots. Generalizing to the case of partial differential equations with constant coefficients (such as the fundamental equations arising in physics), it is natural to seek *plane wave* solutions

$$\exp i(x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n) \tag{4.1}$$

and more general solutions can be obtained by superpositions. In the case of partial differential equations, the characteristic equation would be a polynomial in several variables. For example taking the case of the wave equation

$$u_{tt} - u_{xx} = 0, \tag{4.2}$$

let us substitute the Ansatz (4.1) in the form $\exp i(at - bx)$ into (4.2) and we get

$$a^2 - b^2 = 0. \tag{4.3}$$

Equation (4.3) has infinitely many solutions and indeed two continuous families (λ, λ) and $(\lambda, -\lambda)$. We would now have to take a continuous superposition of the plane waves

$$\exp i\lambda(x + t), \quad \exp i\lambda(x - t)$$

which means we must consider the sum of two integrals

$$\int_{-\infty}^{\infty} f(\lambda)\exp i\lambda(x + t)d\lambda + \int_{-\infty}^{\infty} g(\lambda)\exp i\lambda(x - t)d\lambda. \tag{4.4}$$

We are naturally led to the following

Definition (Fourier transform): Suppose $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a function for which

$$\int_{-\infty}^{\infty} |f(t)|dt \tag{4.5}$$

is finite then the integral

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt$$

is called the Fourier transform of $f(t)$. There are several conventions and we follow the one that is common in PDEs for example see p. 213 of *G. B. Folland, Fourier analysis and its applications.*

Exercises:

1. Let $f(t) = 1$ when $|t| \leq 1$ and $f(t) = 0$ otherwise. Compute the Fourier transform of f. The exercise is easy: One direcly computes:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-it\xi}dt = 2\int_0^1 \cos\xi t dt = \ldots.$$

2. Compute the Fourier transform of the function $f(t)$ given by $f(t) = 1/\sqrt{1-t^2}$ if $|t| < 1$ and $f(t) = 0$ if $|t| \geq 1$. Hint: The Bessel function makes its appearence. You need to recall the integral definition of $J_k(x)$ when $k = 0, 1, 2, \ldots$.

3. Prove the Riemann Lebesgue lemma which states that if $f(t)$ is continuous and the integral (4.5) is finite then $|\widehat{f}(\xi)| \longrightarrow 0$ when $\xi \to \pm\infty$. Hint: Let $\epsilon > 0$ be arbitrary. Select a $K > 0$ such that

$$\int_{\mathbb{R}-[-K,K]} |f(x)|dx < \epsilon/3.$$

4. Compute the Fourier transform of $f_a(t) = 1/(a^2 + t^2)$ where a is a non-zero real number. Note that the cases $a > 0$ and $a < 0$ have to be dealt with separately. Hint: Let $I(\xi)$ be the integral. Find the Laplace transform of $I(\xi)$.

5. Compute the Fourier transform of $\exp(-a|t|)$ where $a > 0$.

6. Looking at the last two examples are you led to conjecture any general result?

7. Calculate the Fourier transform of $f(t) = \sin^2 t/t^2$ using the ideas of exercise (4) above.

8. One can also compute the Fourier transform of $f(t) = \sin t/t$ but a careful justification would have to wait. Why so? However proceed formally and try to arrive at the answer.

9. Try to calculate $f_a * f_b$ where $f_a(t) = a/(\pi(t^2 + a^2))$. Recall the definition of convolution. It is an integral from $-\infty$ to ∞. Don't be too surprised if the computation gets pretty ugly. This example comes up in Probability theory under the name of *Cauchy distribution*.

Fourier transform of the Gaussian This is one of the most important examples in the theory of Fourier transforms and plays a crucial role in probability theory, number theory, quantum mechanics, theory of heat conduction and diffusive processes in general.

Theorem: Suppose $a > 0$. The Fourier transform of $\exp(-at^2)$ is the function

$$\sqrt{\frac{\pi}{a}}\exp(-\xi^2/4a).$$

We have already seen a proof of this in module 1, where we obtained a first order ODE for $I(\xi)$ given by

$$I(\xi) = \int_{-\infty}^{\infty} \exp(-at^2 - it\xi)dt. \tag{4.6}$$

Here we shall give a second proof of this important result. Let us complete the square in (4.6):

$$I(\xi) = e^{-\xi^2/4a} \int_{-\infty}^{\infty} \exp\Big(- a\Big(t + \frac{i\xi}{2a}\Big)^2\Big) dt. \tag{4.7}$$

It is tempting to put $t + \frac{i\xi}{2a} = y$ in (4.7) and proceed formally. We shall refrain from doing so since that is *procedurally wrong!*. Can you explain why? Instead we shall appeal to Cauchy's theorem from complex analysis. Call

$$J = \int_{-\infty}^{\infty} \exp\Big(- a\Big(t + \frac{i\xi}{2a}\Big)^2\Big) dt \tag{4.8}$$

Integrate the analytic function $f(z) = \exp(-az^2)$ along a rectangle with vertices $-R, R, R + (i\xi/2a)$ and $-R + (i\xi/2a)$. Call L_1 the base of the rectangle and L_2 the top side of the rectangle. The vertical sides V_1, V_2 respectively.

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{V_1} f(z)dz + \int_{V_2} f(z)dz = 0. \tag{4.9}$$

We must now let $R \longrightarrow \infty$. Obviously

$$\int_{L_1} f(z)dz \longrightarrow \int_{-\infty}^{\infty} \exp(-at^2)dt = \frac{\sqrt{\pi}}{\sqrt{a}}$$

Also taking into account the direction along L_2,

$$\int_{L_2} f(z)dz \longrightarrow \int_{-\infty}^{\infty} \exp -a\Big(t + \frac{i\xi}{2a}\Big)^2 dt = -J$$

Check that $\int_{V_1} f(z)dz$ and $\int_{V_2} f(z)dz$ individually go to zero as $R \longrightarrow \infty$ and complete the proof of the theorem.

10. Review the earlier procedure for calculating $I(\xi)$ via ODEs.

11. Compute the Fourier transform of $x^2 \exp(-ax^2)$ and more generally $x^{2k} \exp(-ax^2)$.