Fourier Analysis and its Applications

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16 Borel's theorem and beyond

Some ramifications: you may have suspected that the result of Weyl has some connections with probability theory in view of the fact that the limit of the Cesaro averages gives the *expectation* of the *random variable* $f(x)$. This together with the uniform distribution lurking at the background may lead to the belief that Weyl's theorem is merely a special case of a more general phenomena.

This is indeed the case as we now demonstrate with some examples.

Normal numbers and Borel's theorem We first define the notion of a *normal number*. For simplicity we shall look at numbers in $[0, 1]$ and each $x \in [0, 1]$ can be written in *decimal form*:

$$x = 0.a_1a_2a_3\ldots.$$

where $a_1, a_2, a_3, \ldots$ are the digits of the number x. Let us pick a digit say 7 and ask how many times does it appear in the string $a_1, a_2, \ldots, a_n$. Say it appears k_n times. We now consider the ratio k_n/n and ask whether

$$\lim_{n\to\infty} \frac{k_n}{n}$$

exists and if so what is the value. This is the *asymptotic relative frequency* of the occurrence of the digit 7. We say that the number x is normal (with respect to the decimal system) if all digits appear with the *same* relative frequency namely 1/10.

Theorem (Normal numbers and Borel's theorem): The set of numbers in $[0, 1]$ that are not normal has Lebesgue measure zero.

Of course there is nothing special about the decimal system. The result is true for any radix. To bring out the similarity with Weyl's theorem let us define a map $T : [0, 1] \longrightarrow [0, 1]$ by

$$Tx = \{10x\} = 0.a_2a_3a_4\ldots$$

With this, let us assume x is irrational. The digit a_2 will be 7 if and only if $Tx \in (7/10, 8/10)$. Let us look at $T(Tx)$:

$$T(Tx) = T(\{10x\}) = T(0.a_2a_3a_4\ldots) = 0.a_3a_4\ldots$$

and $a_3 = 7$ if and only of $T^2x \in (7/10, 8/10)$. So if χ denotes the characteristic function of the interval $(7/10, 8/10)$ then

$$\frac{k_n}{n} = \frac{1}{n}(\chi(x) + \chi(Tx) + \chi(T^2x) + \cdots + \chi(T^{n-1}x))$$

and we are asking whether for almost all $x \in [0, 1]$

$$\lim_{n\to\infty} \frac{1}{n}(\chi(x) + \chi(Tx) + \chi(T^2x) + \cdots + \chi(T^{n-1}x)) = \int_0^1 \chi(x)dx.$$

and more generally for any integrable function $f(x)$,

$$\lim_{n\to\infty} \frac{1}{n}(f(x) + f(Tx) + f(T^2x) + \cdots + f(T^{n-1}x)) = \int_0^1 f(x)dx.$$

for almost all $x \in [0, 1]$.

Strong law of large numbers and Birkhoff's ergodic theorem The above formulation clearly shows that both Weyl's theorem and Borel's theorem are probably special cases of more general phenomena. That is indeed the case and the general phenomenon is encapsulated by the *strong law of large numbers* in probability theory and even beyond as the *Birkhoff's individual ergodic theorem* in dynamical systems. The latter arose from the study of recurrence in differential equations quantifying a famous theorem of *Poincaré*

For these matters consult the books of *Patrick Billingsley* . His measure and probabilty book gives an "elementary" proof of Borel's theorem. Another important reference is ofcourse the classic work of *G. H. Hardy and E. M. Wright* on number theory.

P. Billingsley, Probability and measure, John Wiley and Sons Inc., 2011

P. Billingsley, Ergodic theory and information, John Wiley and Sons Inc., 1965

With these general comments we must close this module and move on to the theory of *Fourier Transforms.*