Fourier Analysis and its Applications

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15 Weyl's equidistribution theorem.

Weyl's theorem sharpens Kronecker's theorem. Suppose $0 < a < b < 1$ then we know that the interval (a, b) contains infinitely many points

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots \tag{3.6}$$

Now let k_n be the number of points in the list

$$\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\} \tag{3.7}$$

that lie in $[a, b]$. Kronecker's theorem says that $k_n > 0$ if n is sufficiently large. The ratio $\frac{k_n}{n}$ is the fraction of numbers in the list (3.7) that lie in $[a, b]$. Weyl's theorem says that

$$\lim_{n\to\infty} \frac{k_n}{n} = b - a \tag{3.8}$$

Suppose $b - a = 1/4$ then (3.8) says that approximately one fourth of the members (3.6) lie in $[a, b]$ in the long term. Thus Weyl's theorem *quantifies* Kronecker's result. We shall state and prove Weyl's theorem following the treatment in *A. Browder, Mathematical Analysis - an introduction, Springer Verlag, 1996.* Observe that if χ is the characteristic function of $[a, b]$ then $\chi(\{j\alpha\}) = 1$ if $\{j\alpha\} \in [a, b]$ and zero otherwise. Hence the number k_n among the list

$$\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\} \tag{3.7}$$

lying inside $[a, b]$ is precisely

$$\chi(\{\alpha\}) + \chi(\{2\alpha\}) + \cdots + \chi(\{n\alpha\})$$

and so

$$\frac{k_n}{n} = \frac{1}{n}(\chi(\{\alpha\}) + \chi(\{2\alpha\}) + \cdots + \chi(\{n\alpha\})) \tag{3.9}$$

This suggests that more generally for any integrable function $f(x)$ we must construct the Cesaro sums

$$L_n(f) = \frac{1}{n}(f(\{\alpha\}) + f(\{2\alpha\}) + \cdots + f(\{n\alpha\})) \tag{3.10}$$

Theorem Let α be an irrational number. Suppose $f(x)$ is bounded and Riemann-integrable on $[0, 1]$ then the Cesaro sums (3.10) converge pointwise to $\int_0^1 f(x)dx$. In particular taking $f(x)$ to be the characteristic function of $[a, b]$,

$$\lim_{n\to\infty} \frac{1}{n}(\chi(\{\alpha\}) + \chi(\{2\alpha\}) + \cdots + \chi(\{n\alpha\})) = b - a \tag{3.11}$$

Before begining the proof of the theorem let us note the following:

(i) Linearity: $L_n(c_1 f_1 + c_2 f_2) = c_1 L_n(f_1) + c_2 L_n(f_2)$

(ii) Monotonocity: If $f \le g$ then $L_n(f) \le L_n(g)$.

Let us verify the theorem for the case of $f(x) = \exp(2\pi i k x)$.

Proof of Weyl's equidistribution theorem: Since $f(\{j\alpha\}) = \exp(2\pi ik\{j\alpha\}) = \exp(2\pi ijk\alpha - 2\pi ik[j\alpha]) = \exp(2\pi ikj\alpha)$, the sum

$$\frac{1}{n}(f(\{\alpha\}) + f(\{2\alpha\}) + \cdots + f(\{n\alpha\})) \tag{3.12}$$

is simply a finite geometric series with common ratio $\exp(2\pi ik\alpha)$. The expression (3.12) in this case is for $k \neq 0$,

$$\frac{\exp(2\pi ik\alpha)}{n}\frac{1-\exp(2\pi ikn\alpha)}{1-\exp(2\pi ik\alpha)} \longrightarrow 0$$

as $n \to \infty$. On the other hand

$$\int_0^1 \exp(2\pi ikx) = 0, \quad k \neq 0.$$

We see that

$$\lim_{n\to\infty}\frac{1}{n}(f(\{\alpha\}) + f(\{2\alpha\}) + \cdots + f(\{n\alpha\})) = \int_0^1 f(x)dx \tag{3.13}$$

Note that if $k = 0$ then both sides of (3.13) are equal to one. The the theorem has been established for $f(x) = \exp(2\pi ikx)$ for $k = 0, 1, 2, \ldots$ and so by linearity it hold whenever $f(x)$ is a trigonometric polynomial.

To go to the next stage of the proof, Now let $f(x)$ be a continuous $2\pi-$periodic function on the real line and $\epsilon > 0$ be arbitrary. We have proved as a corollary to Fejer's theorem that there is a trigonometric polynomial $g(x)$ such that

$$\sup_{\mathbb{R}} |f(x) - g(x)| < \epsilon$$

Since we have now rescaled the variables by introducing the factor 2π in the argument working with $\exp(2\pi ikx)$ rather than $\exp(ikx)$, the above approximation result must be reformulated as follows. Suppose $f(x)$ is a continuous one periodic function on the real line then $f(x)$ can be approximated in sup norn by a finite linear combination of $\exp(2\pi ikx)$ $(k = 0, 1, 2, \ldots)$.

Now let $\epsilon > 0$ be arrbitrary then there is a one-periodic function $g(x)$ such that

$$\sup_{0\leq x\leq 1} |f(x) - g(x)| < \epsilon/3 \tag{3.14}$$

Now using linearity of L_n,

$$|L_n(f) - \int_0^1 f(x)dx| \leq |L_n(f-g)| + |L_n(g) - \int_0^1 g(x)dx| + \\ + \int_0^1 |f(x) - g(x)|dx$$

Because of (3.14) the first and the last summands are each less than $\epsilon/3$. The middle summand

$$|L_n(g) - \int_0^1 g(x)dx|$$

tends to zero since the theorem has been established for all finite linear combinations of $\exp(2\pi ikx)$ $(k = 0, 1, 2, \ldots)$ and there is an $n_0 \in \mathbb{N}$ such that

$$|L_n(g) - \int_0^1 g(x)dx| < \frac{\epsilon}{3}, \quad n > n_0.$$

Hence

$$|L_n(f) - \int_0^1 f(x)dx| < \epsilon, \quad n > n_0.$$

which shows that the theorem holds for all one-periodic continuous functions $f(x)$. We must now pass from continuous one periodic function to Riemann integrable function. We need the following:

Theorem Suppose $f : [0,1] \longrightarrow \mathbb{R}$ is a bounded Riemann integrable function then given any $\epsilon > 0$, there are continuous functions $g, h : [0,1] \longrightarrow \mathbb{R}$ such that

(i) $g(x) \leq f(x) \leq h(x)$ throughout $[0,1]$.

(ii) $g(0) = g(1)$ and $h(0) = h(1)$.

(iii) $\int_0^1 (h(x) - g(x))dx < \epsilon$.

Exercise: Prove the theorem. We shall obviously extend g, h as one-periodic extension and use the density of trigonometric polynomials with period one.
Hints and suggestions for the previous exercise: First we select a partition $\{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$ with respect to which

$$U(f) - L(f) < \frac{\epsilon}{3}$$

where $U(f)$ and $L(f)$ are the upper and lower Riemann sums of f with respect to this partition. Let M_j, m_j be the supremum and infimum of f on the sub-interval $[t_{j-1}, t_j]$ and M, m are the supremum and infimum of f on the entire interval $[0,1]$ so that

$$m \leq m_j \leq M_j \leq M.$$

Now we consider the step function h_0 which takes the value M_j on the interval $[t_{j-1}, t_j)$. Idea is to modify this h_0 suitably to the requisite continuous function h. To do this we cut out a little piece from the interval $[t_{j-1}, t_j)$ and define $h(x) = h_0(x)$ on the closed subinterval:

$$[t_{j-1} + \eta, t_j - \eta] \tag{3.15}$$

Define $h(0) = h(1) = M$. Now we have defined the function h continuously on the union of the subintervals (3.15) as well as the two endpoints $\{0, 1\}$. We can define the function $h(x)$ in any way we please on the intervals $(t_j - \eta, t_j + \eta)$ subject to it being continuous and greater than equal to $f(x)$. But at the same time less than or equal to say $2M$. Think graphically about this and it will be clear how to do this. For example at t_j define it as $2M$ and then linearly on $[t_j - \eta, t_j]$ and $[t_j, t_j + \eta]$.

Similarly one defines $g(x)$ and then

$$\int_0^1 (h(x) - g(x))dx \leq U(f) - L(f) + \sum_j \int_{t_j - \eta}^{t_j + \eta} (h(x) - g(x))dx$$

The first piece on RHS is less than $\epsilon/3$ and the second piece is less than

$$4n\eta(M - m)$$

This in turn will be smaller than $\epsilon/3$ if we choose η sufficiently small. Finally, let $f(x)$ be bounded Riemann integrable and $\epsilon > 0$. Select $g(x)$ and $h(x)$ as in the last theorem.

$$\begin{aligned}
|L_n(f) - \int_0^1 f(x)dx| &\leq |L_n(f) - L_n(g)| + |L_n(g) - \int_0^1 f(x)dx| \\
&\leq |L_n(f) - L_n(g)| + |L_n(g) - \int_0^1 g(x)dx| + \int_0^1 |f(x) - g(x)|dx \\
&\leq |L_n(f) - L_n(g)| + |L_n(g) - \int_0^1 g(x)dx| + \epsilon
\end{aligned}$$

Now there is an n_1 such that the middle term is less than ϵ for all $n > n_1$. So,

$$|L_n(f) - \int_0^1 f(x)dx| \leq 2\epsilon + L_n(f - g), \quad n > n_1.$$

We have used the monotonocity of L_n. We use it again and write

$$L_n(f - g) \leq L_n(h - g) = |L_n(h - g) - \int_0^1 (h(x) - g(x))dx| + \int_0^1 (h(x) - g(x))dx$$

The last piece is less than ϵ and using the fact that we have established the result for one periodic continuous functions we see that there is a $n_2 \in \mathbb{N}$ such that

$$|L_n(h - g) - \int_0^1 (h(x) - g(x))dx| < \epsilon, \quad n > n_2$$

Taking $n_0 = n_1 + n_2$ we see that

$$|L_n(f) - \int_0^1 f(x)dx| < 4\epsilon, \quad n > n_0.$$

Proof of Weyl's theorem is thereby completed.