Fourier Analysis and its Applications

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14 Kronecker's theorem and Weyl's equidistribution theorem.

Kronecker's theorem and Weyl's equidistribution theorem: Let us recall a classical result due to *Leopold Kronecker* sometimes also known as *Dirichlet's theorem*. Suppose α is an irrational number. Consider the sequence of numbers

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots \tag{3.6}$$

where $\{\theta\}$ denotes the fractional part of θ namely $\{\theta\} = \theta - [\theta]$. The numbers in the list are all in $[0, 1]$ and they are all distinct. Suppose

$$\{k\alpha\} = \{l\alpha\}, \quad k < l.$$

Then $(k - l)\alpha = [k\alpha] - [l\alpha]$ which would mean α is a ratio of two integers. Contradiction since α was assumed to be irrational. Thus the sequence (3.6) must have limit points in $[0, 1]$. Kronecker's theorem asserts that the sequence (3.6) is dense in $[0, 1]$. Let us provide a simple proof of this fact.

We begin with the simple observation that it is enough to prove that the set

$$H = \{m + n\alpha \ : \ m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. Because then $H \cap [0, 1)$ would be dense in $[0, 1)$ and our job would be done as soon as we show $H \cap [0, 1)$ is precisely the set of numbers (3.6). Well, it is clear that each $\{n\alpha\}$ lies in $H \cap [0, 1)$. Conversely, if $x \in H \cap [0, 1)$ then

$$x = \{x\} = m + n\alpha = m + [n\alpha] + \{n\alpha\}$$

This shows that $m + [n\alpha] = 0$ and so $x = \{n\alpha\}$.

To show that H is dense in $\mathbb{R}$ we proceed as follows. Observe that H is a subgroup of $\mathbb{R}$ and H is not cyclic. The result now follows from a more general result in the next slide. We prove generally the following result:

Theorem: If a subgroup of $\mathbb{R}$ is not cyclic then it is dense in $\mathbb{R}$. Now, our H in the previous slide is not cyclic (why?) and hence it is dense in $\mathbb{R}$.

Proof: Suppose H is a non-cyclic subgroup of $\mathbb{R}$ so that in particular $H \neq \{0\}$. Take an $x \in H$ with $x \neq 0$ and $-x \in H$ which means H must contain positive elements. Let

$$\mu = \inf\{y \in H \ : \ y > 0\}$$

We show that if $\mu > 0$ then H is cyclic. Well, if $\mu \in H$ then $H \cap (\mu, 2\mu)$ is empty because if $y \in H \cap (\mu, 2\mu)$ then $y - \mu \in H$ and $0 < y - \mu < \mu$ which contradicts the definition of μ. Similarly one sees that $H \cap (j\mu, (j + 1)\mu)$ is empty for every $j \in \mathbb{Z}$ and H is the cyclic group generated by μ. Our hypothesis precludes this. Now suppose $\mu > 0$ and $\mu \notin H$. Pick a sequence x_n of positive elements in H converging to μ. We may by passing to a subsequence assume that x_n is strictly decreasing. Take $\epsilon = \mu/2$, then there is an n_0 such that

$$|x_n - \mu| < \mu/2, \quad n \geq n_0$$

Pick $m > n > n_0$ and we have

$$\mu/2 < x_m < x_n < 3\mu/2$$

so that $0 < x_n - x_m < \mu/2$. But since $x_n - x_m \in H$ we have a contradiction.

We next take up *the case* $\mu = 0$. We have to show that every interval (a, b) contains an element of H. Well since there is a sequence of positive elements $y_n \in H$ converging to 0, pick a $y_n \in H$ such that

$$0 < y_n < \frac{1}{3}(b - a).$$

Now there must be a least natural number k such that $ky_n > a$ which means $(k-1)y_n \leq a$ or in other words $ky_n \leq y_n + a$. Thus,

$$a < ky_n \leq a + y_n < a + \frac{1}{3}(b - a) < b$$

and we see that $ky_n \in H \cap (a, b)$. The proof is complete.

Kronecker's theorem. Some exercises.

(1) Show that the sequence $\sin 1, \sin 2, \sin 3, \ldots,$ has least upper bound 1.

(2) Suppose a continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ has two periods λ and μ that are linearly independent over the rationals then f is constant.

(3) Discuss the periodic solutions of the differential equation

$$y'' + y = \sin \sqrt{2}t$$

You may have heard of Lissajeous figures in connection with coupled harmonic oscillators in physics. Can you relate this to Kronecker's theorem? When the frequencies are incommensurate what you see on the screen of the oscillioscope is a dimly lit rectangle! why is this happening? Weyl's theorem sharpens Kronecker's theorem. Suppose $0 < a < b < 1$ then we know that the interval (a, b) contains infinitely many points

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots \tag{3.6}$$

Now let k_n be the number of points in the list

$$\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\} \tag{3.7}$$

that lie in $[a, b]$. Kronecker's theorem says that $k_n > 0$ if n is sufficiently large. The ratio $\frac{k_n}{n}$ is the fraction of numbers in the list (3.7) that lie in $[a, b]$. Weyl's theorem says that

$$\lim_{n\to\infty} \frac{k_n}{n} = b - a \tag{3.8}$$

Suppose $b - a = 1/4$ then (3.8) says that approximately one fourth of the members (3.6) lie in $[a, b]$ in the long term.